Generalized numerical semigroups up to permutations of coordinates and some related procedures

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C, G. Failla, F. Navarra, Generalized numerical semigroups up to permutations of coordinates, preprint, 2024.

Generalized Numerical Semigroups

Let $\mathbb{N}$ be the set of non-negative integers.

G. Failla, C. Peterson, R. Utano: Algorithms and basic asymptotics for generalized numerical semigroups in $\mathbb{N}^d$, Semigroup Forum **92**(2), 460–473 (2016).

$S \subseteq \mathbb{N}^d$ is a Generalized Numerical Semigroup (GNS) if:

- For all $\mathbf{x}, \mathbf{y} \in S$ then $\mathbf{x} + \mathbf{y} \in S$.
- $\mathbf{0} = (0, \ldots, 0) \in S$.
- $\mathbb{N}^d \setminus S$ is a finite set.

First definitions:

1. $\mathrm{H}(S) = \mathbb{N}^d \setminus S$ the set of gaps (or holes) of S.
2. $\mathrm{g}(S) = |\mathbb{N}^d \setminus S|$, genus of S

If $d = 1$ then S is a numerical semigroup.

Some useful facts and notions:

Every GNS is a finitely generated submonoid of $\mathbb{N}^d$

Definition

A total order, $\preceq$, on the elements of $\mathbb{N}^d$ is called a relaxed monomial order if it satisfies:

i) If $\mathbf{v}, \mathbf{w} \in \mathbb{N}^d$ and if $\mathbf{v} \preceq \mathbf{w}$ then $\mathbf{v} \preceq \mathbf{w} + \mathbf{u}$ for any $\mathbf{u} \in \mathbb{N}^d$.

ii) If $\mathbf{v} \in \mathbb{N}^d$ and $\mathbf{v} \neq \mathbf{0}$ then $\mathbf{0} \prec \mathbf{v}$.

All monomial orders are relaxed monomial orders (e.g. lexicographic, graded lexicographic,...)

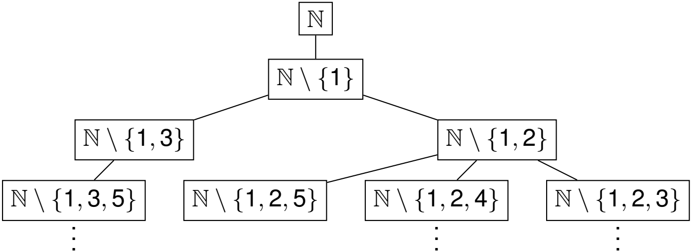
Let $S \subseteq \mathbb{N}^d$ be a GNS and $\preceq$ be a relaxed monomial order in $\mathbb{N}^d$, define:

- $\mathbf{F}_{\preceq}(S) = \max_{\preceq}(\mathrm{H}(S))$
- $\mathbf{U}_{\preceq}(S) = \{\mathbf{x} \text{ minimal generator of } S \mid \mathbf{F}_{\preceq}(S) \prec \mathbf{x}\}$

Some ideas that inspired this work

The tree of numerical semigroups up to a give genus.

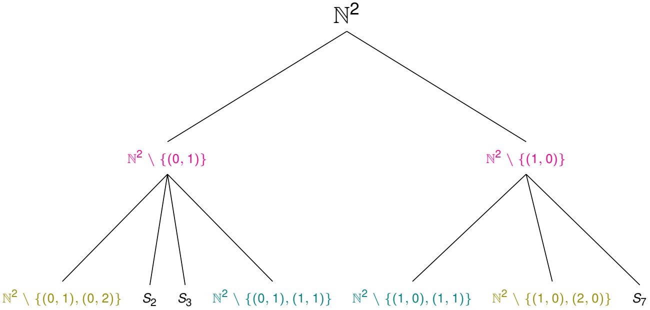
M. Bras-Amorós. Bounds on the number of numerical semigroups of a given genus. *Journal of Pure and Applied Algebra*, 213(6):997–1001, 2009.



Remark: This tree contains all submonoids of $\mathbb{N}$ up to isomorphism.

This construction has been generalized for GNSs in $\mathbb{N}^d$, $d > 1$.

Building a tree for GNSs in $\mathbb{N}^2$, up to a given genus:
(S, T) is an edge if $T = S \cup \{\mathbf{F}_{\preceq}(S)\}$



Up to permutations, some semigroups are redundant.

$$\mathbb{N}^2 \setminus \{(0,1),(1,1)\} = \langle (0,2),(0,3),(1,0),(2,1) \rangle$$
$$\mathbb{N}^2 \setminus \{(1,0),(1,1)\} = \langle (2,0),(3,0),(0,1),(1,2) \rangle$$

These semigroups are isomorphic.

Permutations and isomorphisms

Let $d \in \mathbb{N}$. We denote by $\mathbf{e}_1, \ldots, \mathbf{e}_d$ the standard basis vectors of $\mathbb{N}^d$.

Notations:

- P_d is the set of permutations on $\{1, \ldots, d\}$.
- Let $\sigma \in \mathrm{P}_d$. If $\mathbf{x} = \sum_i^d x_i \mathbf{e}_i$, we define $\sigma(\mathbf{x}) = \sum_i^d x_i \mathbf{e}_{\sigma(i)}$.
- If $A \subseteq \mathbb{N}^d$ we define $\sigma(A) = \{\sigma(\mathbf{a}) \mid \mathbf{a} \in A\}$.

Proposition

Let S be a submonoid of $\mathbb{N}^d$ and $\sigma \in \mathrm{P}_d$. Let

$$f_\sigma : S \to \sigma(S) \text{ defined by } f_\sigma(\mathbf{s}) = \sigma(\mathbf{s})$$

Then f_σ is an isomorphism.

Moreover if S is a GNS, then $\mathrm{H}(\sigma(S)) = \sigma(\mathrm{H}(S))$.

Isomorphisms between GNS in $\mathbb{N}^d$.

Theorem

Let S, T be GNSs in $\mathbb{N}^d$ and suppose there exists an isomorphism $f : S \to T$. Then there exists $\sigma \in \mathrm{P}_d$ such that $T = \sigma(S)$.

Let g, d positive integers, denote:

- $\mathcal{S}_d = \{S \subseteq \mathbb{N}^d \mid S \text{ is a GNS}\}$.
- $\mathcal{S}_{g,d} = \{S \in \mathcal{S}_d \mid \mathrm{g}(S) = g\}$.

We introduce in $\mathcal{S}_d$ the equivalence relation

$$S \simeq T \Leftrightarrow \text{ there exists } \sigma \in \mathrm{P}_d \text{ such that } T = \sigma(S)$$

If $S \in \mathcal{S}_d$, then $[S]_\simeq$ is the set of GNSs in $\mathbb{N}^d$ isomorphic to S.

Observe that if $\mathrm{g}(S) = g$, then $[S]_\simeq \subseteq \mathcal{S}_{g,d}$.

Representatives

Let $\preceq$ be a relaxed monomial order on $\mathbb{N}^d$ and $S, S' \in \mathcal{S}_{g,d}$. Assume

- $\mathrm{H}(S) = \{\mathbf{h}_1 \prec \mathbf{h}_2 \prec \cdots \prec \mathbf{h}_g\}$
- $\mathrm{H}(S') = \{\mathbf{h}'_1 \prec \mathbf{h}'_2 \prec \cdots \prec \mathbf{h}'_g\}$

If $S \neq S'$, define:

$$\mathrm{r}_{\prec}(S, S') = \min\{i \in \{1, \ldots, g\} \mid \mathbf{h}_i \neq \mathbf{h}'_i\}$$

Define the following total order relation on the set $\mathcal{S}_{g,d}$:

$$S \prec_{\mathrm{R}} S' \quad \iff \quad \mathbf{h}_{\mathrm{r}_{\prec}(S,S')} \prec \mathbf{h}'_{\mathrm{r}_{\prec}(S,S')}.$$

For $S \in \mathcal{S}_{g,d}$, denote

$$\mathrm{R}_{\preceq}(S) = \min_{\preceq_{\mathrm{R}}}([S]_{\simeq})$$

and we call it the representative of S with respect to $\preceq$.

Example

Consider the following GNS:

$S = \mathbb{N}^3 \setminus \{(0,1,0), (0,2,0), (0,3,0), (1,0,0), (1,1,0), (2,0,0), (3,0,0)\}.$

$\mathrm{P}_3 = \{id, \sigma_2 := (12), \sigma_3 := (13), \sigma_4 := (23), \sigma_5 := (123), \sigma_6 := (132)\}.$

- $id(S) = \sigma_2(S) = S.$
- $\sigma_3(S) = \sigma_5(S) = S_2 :=$
  $\mathbb{N}^3 \setminus \{(0,0,1), (0,0,2), (0,0,3), (0,1,1), (0,1,0), (0,2,0), (0,3,0)\}$
- $\sigma_4(S) = \sigma_6(S) = S_3 :=$
  $\mathbb{N}^3 \setminus \{(0,0,1), (0,0,2), (0,0,3), (1,0,0), (1,0,1), (2,0,0), (3,0,0)\}$

If $\preceq$ is the lexicographic order, then $[S]_{\simeq} = \{S_2 \preceq_{\mathrm{R}} S_3 \preceq_{\mathrm{R}} S\}$.

In particular, $\mathrm{R}_{\preceq}(S) = S_2$.

Example

Consider the following GNS:

$S = \mathbb{N}^3 \setminus \{(0,1,0),(0,2,0),(0,3,0),(1,0,0),(1,1,0),(2,0,0),(3,0,0)\}.$

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In particular, $\mathrm{R}_{\preceq}(S) = S_2$.

Compute representatives of given genus

Let $\preceq$ be a relaxed monomial order in $\mathbb{N}^d$. Define

- $\mathrm{R}_{\preceq}(\mathcal{S}_d) = \{\mathrm{R}_{\preceq}(S) \mid S \subseteq \mathbb{N}^d \text{ and } S \text{ is a GNS}\}$
- $\mathrm{R}_{\preceq}(\mathcal{S}_{g,d}) = \{\mathrm{R}_{\preceq}(S) \mid S \subseteq \mathbb{N}^d \text{ and } S \text{ is a GNS with } \mathrm{g}(S) = g\}$

Consider the transform:

$$\mathcal{J}_{d,\preceq} : \mathcal{S}_d \setminus \{\mathbb{N}^d\} \to \mathcal{S}_d \quad \text{defined by} \quad S \mapsto S \cup \{\mathbf{F}_{\preceq}(S)\}$$

Is it possible to restrict the transform to $\mathrm{R}_{\preceq}(\mathcal{S}_d)$?

Lemma

Let $S \in \mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$ for positive integers g, d and $\preceq$ be a relaxed monomial order. Then $S \cup \{\mathbf{F}_{\preceq}(S)\} \in \mathrm{R}_{\preceq}(\mathcal{S}_{g-1,d})$.

Let $\mathcal{T}^{\mathrm{R}}_{d,\preceq}$ be the oriented graph such that

- $\mathrm{R}_{\preceq}(\mathcal{S}_d)$ is the set of vertices.
- (S, T) is an edge if $T = S \cup \{\mathbf{F}_{\preceq}(S)\}$

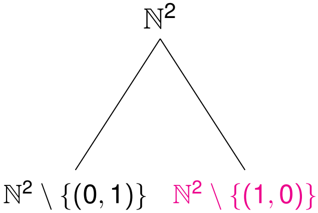
Theorem

Let $\preceq$ be a relaxed monomial order in $\mathbb{N}^d$. Then the following holds:

1. *The graph $\mathcal{T}^{\mathrm{R}}_{d,\preceq}$ is a tree whose root is $\mathbb{N}^d$.*
2. *$S \in \mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$ if and only if S is a vertex of depth g in $\mathcal{T}^{\mathrm{R}}_{d,\preceq}$.*
3. *For every $S \in \mathrm{R}_{\preceq}(\mathcal{S}_d)$, the children of S in $\mathcal{T}^{\mathrm{R}}_{d,\preceq}$ are the GNSs $S \setminus \{\mathbf{n}\}$ such that*

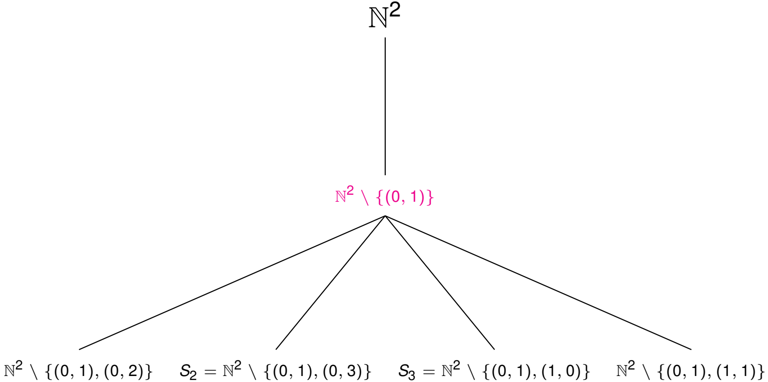
$$\mathbf{n} \in \mathbf{U}_{\preceq}(S) \quad \textit{and} \quad S \setminus \{\mathbf{n}\} \in \mathrm{R}_{\preceq}(\mathcal{S}_d)$$

Building the tree of GNSs in $\mathbb{N}^2$, with $\preceq$ lexicographic order.



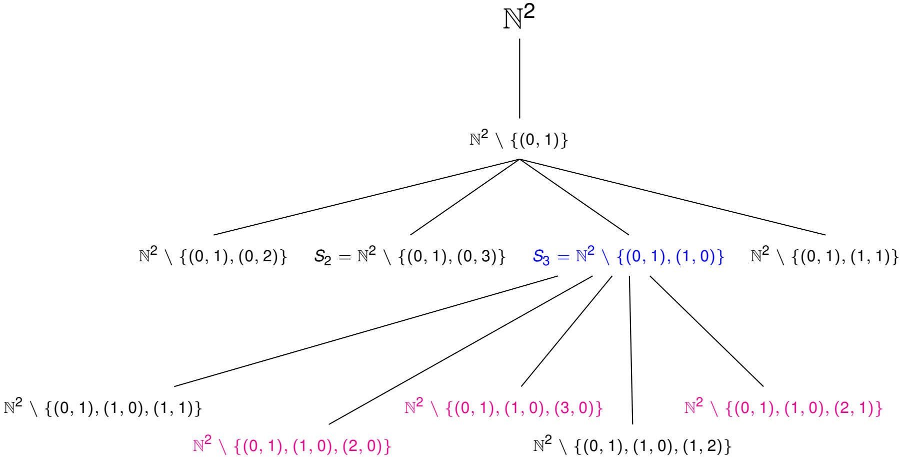
$\mathbb{N}^2 \setminus \{(1,0)\}$ is not a representative.

Removing generators of $\mathbf{U}_{\preceq}(S)$ from $S = \mathbb{N}^2 \setminus \{(0,1)\}$.

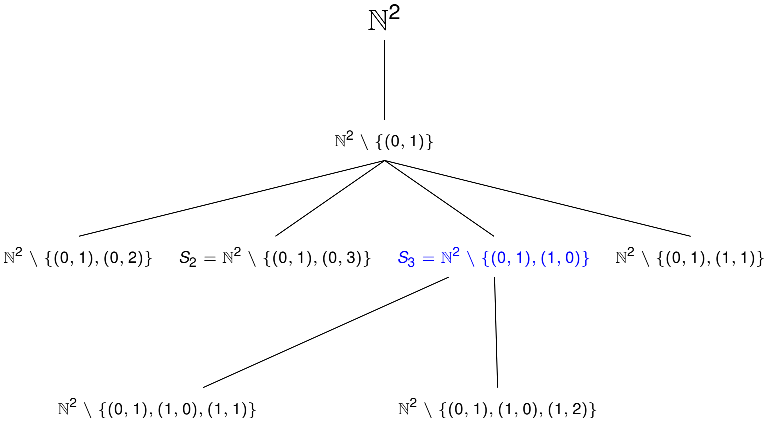


In this case, for all $\mathbf{n} \in \mathbf{U}_{\preceq}(S)$ we have $S \setminus \{\mathbf{n}\} \in \mathrm{R}_{\preceq}(\mathcal{S}_d)$.

Removing generators of $\mathbf{U}_{\preceq}(S)$ from $S = \mathbb{N}^2 \setminus \{(0,1),(1,0)\}$.



Removing non representatives.



Question: For $T \in \mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$ and $\mathbf{n}$ is a minimal generator of T, when $T \setminus \{\mathbf{n}\} \in \mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$?

A sufficient condition:

Proposition

Let g, d be positive integers, $\preceq$ be a relaxed monomial order and $T \in \mathrm{R}_{\preceq}(\mathcal{S}_{g-1,d})$.
Let $\mathbf{n}$ be a minimal generator of T such that $\mathbf{n} \preceq \sigma(\mathbf{n})$ for all $\sigma \in \mathrm{P}_d$.
Then $T \setminus \{\mathbf{n}\} \in \mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$.

An observation about another construction:

Let $\preceq$ be a relaxed monomial order in $\mathbb{N}^d$. If $S \subseteq \mathbb{N}^d$ is a GNS, define $\mathbf{m}_{\preceq}(S) = \min_{\preceq}(S \setminus \{0\})$.

C., Manuel Delgado, and Pedro A. García-Sánchez. Algorithms for generalized numerical semigroups, *Journal of Algebra and Its Applications*, 20(05):2150079, 2021.

It is shown how to arrange the set $\mathcal{S}_{g,d}$ in a tree. This construction is related to the transform

$$\mathcal{O}_{\preceq} : \mathcal{S}_{g,d} \to \mathcal{S}_{g,d} \quad \text{with} \quad S \mapsto (S \cup \{\mathbf{F}_{\preceq}(S)\}) \setminus \{\mathbf{m}_{\preceq}(S)\}$$

Is it possible to restrict the transform to $\mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$?

Let $S \in \mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$

- If $\preceq$ is the lexicographic order, then $\mathcal{O}_{\preceq}(S) \in \mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$ for all S.
- If $\preceq$ is the graded lexicographic order, then $\mathcal{O}_{\preceq}(S) \notin \mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$ for some S.

Example

Let $\preceq$ be the graded lexicographic order in $\mathbb{N}^2$.

$$S = \mathbb{N}^2 \setminus \{(0,1),(1,0),(0,2),(1,1),(2,0),\\ (0,3),(1,2),(3,0),(4,0),(4,1)\}$$

$S \in \mathrm{R}_{\preceq}(\mathcal{S}_{10,2})$, but $\mathcal{O}_{\preceq}(S) \notin \mathrm{R}_{\preceq}(\mathcal{S}_{10,2})$. In fact:

$$\mathcal{O}_{\preceq}(S) = S \cup \{(4,1)\} \setminus \{(2,1)\} = \\ \mathbb{N}^2 \setminus \{(0,1),(1,0),(0,2),(1,1),(2,0),\\ (0,3),(1,2),(2,1),(3,0),(4,0)\}$$

Question: For what kind of relaxed monomial order is it possible to restrict $\mathcal{O}_{\preceq}$ to $\mathrm{R}_{\preceq}(\mathcal{S}_{g,d})$?

M. Delgado, P. A. García-Sánchez, J. Morais, NumericalSgps, A package for numerical semigroups, Version 1.3.1 dev (2023), Refereed GAP package, https://gap-packages.github.io/numericalsgps.

Table: $N_{g,d} = |\,\mathrm{R}_{\preceq}(\mathcal{S}_{g,d})|$ (compared with $n_{g,d} = |\mathcal{S}_{g,d}|$ for $d = 2, 3$)

g	$n_{g,2}$	$N_{g,2}$	$n_{g,3}$	$N_{g,3}$	$N_{g,4}$	$N_{g,5}$	$N_{g,6}$
1	2	1	3	1	1	1	1
2	7	4	15	4	4	4	4
3	23	12	67	15	15	15	15
4	71	37	292	59	64	64	64
5	210	107	1215	224	270	277	277
6	638	323	5075	903	1254	1344	1355
7	1894	953	20936	3611	5945	6810	
8	5570	2798	85842	14603	29132	36536	
9	16220	8128	349731	58954			
10	46898	23486	1418323	237956			
11	134856	67477					
12	386354	193285					
13	1102980	551628					
14	3137592	1569107					

Theorem

Let g be a positive integer. Then $N_{g,d} = N_{g,g}$ for all $d \geq g$.

Equivariant GNSs

We say that $S \subseteq \mathbb{N}^d$ is an equivariant GNS if $[S]_{\simeq} = \{S\}$, that is:

$$\sigma(S) = S \text{ for all } \sigma \in \mathrm{P}_d$$

Equivalently, S is equivariant if and only if $\sigma(\mathrm{H}(S)) = \mathrm{H}(S)$ for all $\sigma \in \mathrm{P}_d$.

Example

The following GNS is equivariant:

$S = \mathbb{N}^3 \setminus \{(0,0,1),(0,0,2),(0,1,0),(0,2,0),(1,0,0),(1,1,1),(2,0,0)\}$.

Consider $\sigma = (2,3), \tau = (13) \in \mathrm{P}_3$ and the following:

$S' = \mathbb{N}^3 \setminus \{(0,0,1),(0,0,2),(0,0,3),(0,1,0),(0,1,1),(0,2,0),(0,3,0)\}$.

$\sigma(S') = S'$ but S' is not equivariant since $\tau(S') \neq S'$

A tree containing all equivariant GNSs

For $\mathbf{x} \in \mathbb{N}^d$, define $\operatorname{orb}(\mathbf{x}) = \{\sigma(\mathbf{x}) \in \mathbb{N}^d \mid \sigma \in \mathrm{P}_d\}$.

Define $\mathcal{G}^d_{\preceq}$ be the oriented graph such that

- The set $\mathcal{A}_d$ of all equivariant GNSs in $\mathbb{N}^d$ is the set of vertices.
- (S, T) is an edge if $T = S \cup \operatorname{orb}(\mathbf{F}_{\preceq}(S))$

Theorem

Let $\preceq$ be a relaxed monomial order in $\mathbb{N}^d$. Then the graph $\mathcal{G}^d_{\preceq}$ is a tree whose root is $\mathbb{N}^d$.
Moreover, let $T \in \mathcal{A}_d$ and consider the disjoint union

$$\mathbf{U}_{\preceq}(T) = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_r$$

such that if $\mathbf{x} \in A_i$ and $\mathbf{y} \in A_j$ with $i \neq j$ then $\mathbf{x} \notin \operatorname{orb}(\mathbf{y})$.
For all $i \in \{1, \ldots, r\}$ select one element $\mathbf{x}_i \in A_i$.
Then the children of T in $\mathcal{G}^d_{\preceq}$ are the semigroups $T \setminus \operatorname{orb}(\mathbf{x}_i)$ for $i \in \{1, \ldots, r\}$.

Building the tree of equivariant GNSs in $\mathbb{N}^2$.

- $\mathbb{N}^2$
  - $\mathbb{N}^2 \setminus \{(0,1),(1,0)\}$
    - $\mathbb{N}^2 \setminus \{(0,1),(1,0),(0,2),(2,0)\}$
    - $\mathbb{N}^2 \setminus \{(0,1),(1,0),(0,3),(3,0)\}$
    - $\mathbb{N}^2 \setminus \{(0,1),(1,0),(1,1)\}$
    - $\mathbb{N}^2 \setminus \{(0,1),(1,0),(1,2),(2,1)\}$

$$\mathbb{N}^2 \setminus \{(0,1),(1,0)\} = \langle (0,2),(2,0),(0,3),(3,0),(1,1),(1,2),(2,1) \rangle$$

Possible future investigations

In the context of numerical semigroups and affine semigroup there are different graph-tree construction to produce families of semigroups with prescribed properties.

M. Bernardini, W. Tenório, G. Tizziotti: The corner element of generalized numerical semigroups. Results Math. 77, no.141, 2022.

J. I. García-Garcá, D. Marín-Aragón, D., A. Sánchez-Loureiro, A. Vigneron Tenorio: Some Properties of Affine C-semigroups. Results Math, 79, no.52, 2024.

Investigate how these procedures can be refined in order to produce only one generalized numerical semigroup in every class of isomorphism.

Thank for your attention