Arf Numerical Semigroups with Prime Power Multiplicity

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Notations

$\mathbb{Z}$: **the set of integers,** $\mathbb{N}$: **the set of positive integers,** $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $A \subseteq \mathbb{N}_0$. **The submonoid of** $\mathbb{N}_0$ **generated by** A **is**

$$\langle A \rangle = \left\{ \sum_{i=1}^{r} x_i a_i : r \in \mathbb{N}, x_1, \ldots, x_r \in \mathbb{N}_0, a_1, \ldots, a_r \in A \right\}$$

A submonoid A **of** $\mathbb{N}_0$ **is called** a **numerical semigroup** **if its complement in** $\mathbb{N}_0$ **is finite.**

A submonoid $\langle A \rangle$ **is a numerical semigroup if and only if** $\text{g.c.d}(A) = 1$.

If $A = \{a_1, \ldots, a_e\}$, **we write** $\langle A \rangle = \langle a_1, \ldots, a_e \rangle$.

For a numerical semigroup S**,**

$m(S)$**: the multiplicity of** S**;**

$e(S)$**: the embedding dimension of** S**;**

$f(S)$**: the Frobenius number of** S**;**

$c(S)$**: the conductor of** S**;**

$g(S)$**: the genus of** S**;**

$R(S)$: **the ratio of** S.

S : **a numerical semigroup,** $m(S) = m$, $e(S) = e$, $f(S) = f$, $c(S) = c$, $g(S) = g$, **and** $R(S) = R$.

Customary notation for $S \neq \mathbb{N}_0$ **with conductor** c**:**

$$S = \{s_0 = 0, s_1 = m, s_2, \ldots, s_{n-1}, s_n = c \rightarrow\}$$

where $s_{i-1} < s_i$ **for** $1 \leq i \leq n$. **The elements** $s_0 = 0, s_1 = m, s_2, \ldots, s_{n-1}$ **are called small elements of** S**. The number of small elements is** $n = n(S) = |S \cap \{0, 1, \ldots, f\}|$.

It is easily observed that $g+n = c$, $n \leq g$ **and thus** $2n \leq c \leq 2g$**.**

Apéry Sets

For $a \in S \setminus \{0\}$, **the Apéry set of S with respect to** a **is defined as**

$$\mathrm{Ap}(S, a) = \{s \in S : s - a \notin S\}.$$

We have

$$\mathrm{Ap}(S, a) = \{w(0)=0, w(1), . . . , w(a - 1)\}$$

where $w(i) = \min\{x \in S : x \equiv i \pmod{a}\}$, $0 \leq i \leq a$-1.

$S = \langle a, w(1), . . . , w(a - 1)\rangle$

$f(S) = \max(Ap(S, a)) - a.$

Taking $a = m$**, we see that** $S = \langle m, w(1), . . . , w(m - 1)\rangle$. **Thus** $e \leq m$.

Quotient of a Semigroup

For a numerical semigroup S and a positive integer d, the quotient $\frac{S}{d}$ of S by d is definned as

$$\frac{S}{d} = \{x \in \mathbb{N}_0 ; dx \in S\}.$$

It is easy to see that $\frac{S}{d}$ is a numerical semigroup containing S, and $\frac{S}{d} = \mathbb{N}_0$ if, and only if $d \in S$. It is also easy to see that if $a \in S \setminus \{0\}$ and if d is a divisor of a, then

$$\mathrm{Ap}\left(\frac{S}{d}, \frac{a}{d}\right) = \left\{\frac{w}{d} : w \in \mathrm{Ap}(S,a) \text{ , } d \text{ is a divisor of } w\right\}.$$

Arf Semigroups

A numerical semigroup S is called an Arf numerical semigroup, or simply an Arf semigroup if the following condition is satisfied:

$$x, y, z \in S;\ x \geq y \geq z \ \Rightarrow\ x + y - z \in S. \qquad \text{(the Arf condition)}$$

$\mathbb{N}_0$ is an Arf semigroup.

$\{0, g+1\rightarrow\}$ **is an Arf semigroup for any $g \in \mathbb{N}$.**

$S = \langle 4, 7\rangle$ **is not an Arf semigroup:** $7 + 7 - 4 \notin S$.

Note that a numerical semigroup S is an Arf semigroup if and only if the Arf condition is satisfied by the small elements of S.

Every Arf semigroup is of maximal embedding dimension: $e(S) = m(S)$

Thus if S is an Arf semigroup, then $\{\mathrm{Ap}(S,m) \setminus \{0\}) \cup \{m\} = \{m, w(1), \ldots, w(m-1)\}$, **is the minimal set of generators of S.**

$\mathscr{S}_{ARF}(m, c)$: **the set of Arf semigroups with multiplicity** m **and conductor** c**.**

$N_{ARF}(m,c)$ **: the number of Arf semigroups with multiplicity** m **and conductor** c**.**

In a recent paper we proved that $N_{ARF}(p,c) = N_{ARF}(p,c+p)$ **if** p **is prime and** $c > 2p$ **.**

Thus $N_{ARF}(p,c)$ **is (eventually) a constant function when restricted to congruence classes modulo** p**.**

In the same paper we had noticed that the above property holds also for $N_{ARF}(m,c)$ **with composite** m**, not for all but some congruence classes modulo** m**.**

In the present work, we prove that

$$N_{ARF}(p^n, c) = N_{ARF}(p^n, c+p^n)$$

if p **is prime,** $n \in \mathbb{N}$**,** $c > 2p^n$ **and**

$$c \equiv (tp^{n-1}+1) \pmod{p^n}, 1 \leq t \leq p\text{-}1.$$

That is, $N_{ARF}(p^n, c)$ **is (eventually) a constant function when restricted to congruence classes of** $(tp^{n-1}+1)$ **modulo** p^n **.**

Theorem. **Let** $p,n,c \in \mathbb{N}$**, where** p **is prime,** $c \equiv (t\,p^{n-1}+1)(\mathrm{mod}\ p^n)$, $1 \leq t \leq p-1$**, and** $c > 2p^n$ **.** **Then**

$$\mathscr{S}_{ARF}(p^n, c + p^n) = \{(p^n + S) \cup \{0\} : S \in \mathscr{S}_{ARF}(p^n, c)\}$$

and thus

$$N_{ARF}(p^n, c + p^n) = N_{ARF}(p^n, c).$$

Lemma 1. **Let** S **be an Arf numerical semigroup and** $s \in S$. **If** $s+1 \in S$ **, then** $s+k \in S$ **for all** $k \in \mathbb{N}_0$ **and thus** $c \leq s$**.**

Lemma 2. **If** S **is an Arf numerical semigroup and** $d \in \mathbb{N}$ **, then** $\frac{S}{d}$ **is an Arf numerical semi-group. Moreover, if** d **is a divisor of** $m(S)$**, then** $m\left(\frac{S}{d}\right) = \frac{m(S)}{\mathrm{d}}$.

Lemma 3. **Let** S **be an Arf numerical semigroup with multiplicity** m **and conductor** c. **For any** $s \in S \backslash \{0\}$, $(s+S) \cup \{0\}$ **is an Arf numerical semigroup with multiplicity** s **and conductor** $c+s$**.**

Lemma 4. **Let** S **be an Arf numerical semigroup with multiplicity** m **and conductor** c. **Then** $-m+(S\backslash\{0\})$ **is an Arf numerical semigroup with multiplicity** $s_2 - m$ **and conductor** $c-m$**, where** s_2 **is the third small element of** S**.**

Lemma 1. **Let S be an Arf numerical semigroup and $s \in S$. If $s+1 \in S$, then $s+k \in S$ for all $k \in \mathbb{N}_0$ and thus $c \leq s$.**

Lemma 2. **If S is an Arf numerical semigroup and $d \in \mathbb{N}$, then $\frac{S}{d}$ is an Arf numerical semi-group. Moreover, if d is a divisor of $m(S)$, then $m\left(\frac{S}{d}\right) = \frac{m(S)}{d}$.**

Lemma 3. **Let S be an Arf numerical semigroup with multiplicity m and conductor c. For any $s \in S \backslash \{0\}$, $(s+S) \cup \{0\}$ is an Arf numerical semigroup with multiplicity s and conductor $c+s$.**

Lemma 4. **Let S be an Arf numerical semigroup with multiplicity m and conductor c. Then $-m+(S \backslash \{0\})$ is an Arf numerical semigroup with multiplicity $s_2 - m$ and conductor $c-m$, where s_2 is the third small element of S.**

Corollary. **If the third small element of an Arf numerical semigroup S is a multiple of $m(S)$, that is, $s_2=2m(S)$, then $-m(S)+(S \backslash \{0\})$ is an Arf numerical semigroup with multiplicity $m(S)$, ratio $R(S) - m(S)$ and conductor $c(S) - m(S)$.**

Lemma 5. **Let S be an Arf numerical semigroup with multiplicity m, ratio R, and con-ductor c, where $\gcd(R,m) = 1$. Let r_{n-1} be the remainder just preceding the last nonzero remainder of R and m in the Euclidean algorithm. Then**

(i) $R \geq c - m + r_{n-1} + 1$, $\qquad$ (ii) $\mathrm{Ap}(S,m) \backslash \{0\} \subset (c - m, \infty)$.

Lemma 6. **Let S be an Arf numerical semigroup with multiplicity p^n, ratio R, and conductor $c \equiv (t\,p^{n-1}+1) \pmod{p^n}$, where $n \geq 2$ and $t \in \{1, 2, . . . , p\text{-}1\}$. Then $\frac{S}{p}$ is an Arf numerical semigroup with multiplicity p^{n-1} and conductor $c\left(\frac{S}{p}\right) = \frac{c+p-1}{p}$, whence**

$$c\left(\frac{S}{p}\right) \equiv (tp^{n-2}+1)(\text{mod}\, p^{n-1}).$$

Moreover, if R is divisible by p, then $\frac{R}{p}$ is the ratio of $\frac{S}{p}$.

Lemma 7. **Let S be an Arf numerical semigroup with multiplicity p^n, ratio R, and conductor $c \equiv (t\,p^{n-1}+1) \pmod{p^n}$, where $t \in \{1, 2, . . . , p\text{-}1\}$. Assume also that $c > 2p^n$. Then**

(*i*) $R \geq c - p^n + 3$, (*ii*) $\text{Ap}(S,p^n)\backslash\{0\} \subset (c\text{-}p^n, \infty)$.

Theorem. **Let** $p,n,c \in \mathbb{N}$**, where** p **is prime,** $c \equiv (t\,p^{n-1}+1) (\mathrm{mod}\ p^n)$, $1 \le t \le p-1$**, and** $c > 2p^n$ **.** **Then**

$$\mathscr{S}_{ARF}(p^n, c + p^n) = \{(p^n + S) \cup \{0\} : S \in \mathscr{S}_{ARF}(p^n,c)\}$$

and thus

$$N_{ARF}(p^n, c + p^n) = N_{ARF}(p^n,c).$$

Proof. **Let** $S \in \mathscr{S}_{ARF}(p^n,c)$. **Applying Lemma 3 with** $s = p^n$ **, we get**

$$\{(p^n + S) \cup \{0\} : S \in \mathscr{S}_{ARF}(p^n,c)\} \subseteq \mathscr{S}_{ARF}(p^n, c+p^n) \ .$$

Now, let $T \in \mathscr{S}_{ARF}(p^n,c+p^n)$. **We have** $R(T) \ge (c + p^n)-p^n+ 3 > 2\,p^n$ **by Lemma 7.**

Therefore the second smallest nonzero element in T is $2p^n$**, and thus** $S = -p^n+(T\setminus\{0\})$ **is an Arf numerical semigroup with multiplicity** p^n **and conductor** $c+p^n-p^n = c$ **by Lemma 4.**

Hence $T = (p^n +S) \cup\{0\})$ **where** $S \in \mathscr{S}_{ARF}(p^n,c)$**. So**

$$\mathscr{S}_{ARF}(p^n, c + p^n) \subseteq \{(p^n + S) \cup \{0\} : S \in \mathscr{S}_{ARF}(p^n,c)\},$$

proving the desired equality. The last assertion is then obvious.

Theorem. **Let** $p,n,c \in \mathbb{N}$**, where** p **is prime,** $c \equiv (t\,p^{n-1}+1) \pmod{p^n}$, $1 \le t \le p-1$**, and** $c > 2p^n$**.**
Then

$$\mathscr{S}_{ARF}(p^n, c + p^n) = \{(p^n + S) \cup \{0\} : S \in \mathscr{S}_{ARF}(p^n, c)\}$$

and thus

$$N_{ARF}(p^n, c + p^n) = N_{ARF}(p^n, c).$$

Corollary. **Notations being as in the theorem,**

$$N_{ARF}(p^n, c + hp^n) = N_{ARF}(p^n, c).$$

for any $h \in \mathbb{N}$**.**

Example 1. **Let** $S \in \mathscr{S}_{ARF}(16,c)$ **where** $c \equiv 9 (\mathrm{mod}\ \ 16)$, **and** $c > 32$. **Then**

The ratio R **of** S **is one of**

$$c-12,\ c-6,\ c-4,\ c-3,\ c-2,\ c.$$

There is only one Arf numerical semigroup with ratio $c-k$ **for** $k \in \{0,2,3,6,12\}$ **and there are** 2 **Arf numerical semigroups with ratio** $c-4$.

For instance, $\mathscr{S}_{ARF}(16,41)$ **consists of the following semigroups:**

$S_1=\{0,16,29,32,35,38,41\rightarrow\}$,

$S_2=\{0,16,32,35,38,41\rightarrow\}$,

$S_3=\{0,16,32,37,39,41\rightarrow\}$, $S_4=\{0,16,32,37,41\rightarrow\}$,

$S_5=\{0,16,32,38,41\rightarrow\}$,

$S_6=\{0,16,32,39,41\rightarrow\}$,

$S_{12}=\{0,16,32,41\rightarrow\}$.

Example 2. **Let** $S \in \mathscr{S}_{ARF}(25,c)$ **where** $c \equiv 6 (\bmod\ 25)$, **and** $c > 50$. **Then**

The ratio R **of** S **is one of**

$$c-22,\ c-20,\ c-17,\ c-15,\ c-14,\ c-13,\ c-12,\ c-10,\ c-9,\ c-8,\ c-10,\ c-9,\ c-8,\ c.$$

For instance, $\mathscr{S}_{ARF}(25,56)$ **consists of the following** 38 **semigroups:**

S_1={ 0,25, 34,43,50,52,54,56→},
S_2={ 0,25,36,47,50,53,56→},
S_3={0,25, 39,50,53,56→},
S_4={0,25,41,44,47,50,53,56→},
S_5={0,25,41,47,50,53,56→},
S_6={0,25,41,50,52,54,56→},
S_7={0,25,41,50,53,56→},
S_8={0,25,41,50,54,56→},
S_9={0,25,41,50,56→},
S_{10}={0,25,42,44,46,48,50,52,54,56→},
S_{11}={0,25, 42,46,48,50,52,54,56→},
S_{12}={0,25,42,46,50,52,54,56→},
S_{13}={0,25,42,46,50,54,56→},
S_{14}={0,25,42,48,50,52,54,56→},
S_{15}={0,25,42,50,52,54,56→},
S_{16}={0,25,42,50,53,56→},
S_{17}={0,25,41,50,54,56→},
S_{18}={0,25,41,50,56→} ,
S_{19}={ 0,25, 43,50,56→},

S_{20}={ 0,25,36,47,50,53,56→},
S_{21}={0,25, 39,50,53,56→},
S_{22}={0,25,41,44,47,50,53,56→},
S_{23}={0,25,41,47,50,53,56→},
S_{24}={0,25,41,50,52,54,56→},
S_{25}={0,25,41,50,53,56→},
S_{26}={0,25,41,50,54,56→},
S_{27}={0,25,41,50,56→},
S_{28}={0,25,42,44,46,48,50,52,54,56→},
S_{29}={0,25, 42,46,48,50,52,54,56→},
S_{30}={0,25,42,46,50,52,54,56→},
S_{31}={0,25,42,46,50,54,56→},
S_{32}={0,25,42,48,50,52,54,56→},
S_{33}={0,25,42,50,52,54,56→},
S_{34}={0,25,42,50,53,56→},
S_{35}={0,25,41,50,54,56→},
S_{36}={0,25,41,50,56→}
S_{37}={0,25,41,50,54,56→},
S_{38}={0,25,41,50,56→}

References

1. C. Arf, *Une interprétation algébrique de la suite ordres de multiplicité d'une branche algébrique*, Proc. London Math. Soc. **20** (1949), 256-287.

2. V. Barucci, D. E. Dobbs, and M. Fontana, *Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains*, Memoirs of the American Mathematical Society **125/598** (1997), 1-77.

3. A. Compillo, J. I. Farran, and C. Munuera, *On the parameters of algebraic geometry codes related to arf semigroups*, IEEE Transactions on Information Theory **46(7)** (2000), 2634-2638.

4. R. Fröberg, C. Gottlieb, and R. Häggkvist, *On numerical semigroups*, Semigroup Forum 3**5**(1987), 63-83.

5. P. A. Garcia-Sánchez, B. A. Heredia, H. İ. Karakaş, and J. C. Rosales, *Parametrizing arf nu-nu-merical semigroups*, J. Algebra and Appl. **16(11)** (2017), 1750209.

6. H. İ. Karakaş, *Arf numerical semigroups with multiplicity* 9 *and* 10, Numerical Semi-groups (eds: V. Barucci and S. Chapman and M. D'Anna and R. Fr□berg). Springer INdAM Series. Springer, Cham. **40** (2020), 163-183.

7. __________ , *Arf numerical semigroups with prime multiplicity*, Semigroup Forum **105** (2022), 478-487.

8. H. İ. Karakaş, S. İlhan, and M. Süer, *Arf numerical semigroups with multiplicity* 11 *and* 13, Turk. J. Masth **46** (2022), 1446-1458.

9. Michal Lason, *On the relation between betti number of an arf semigroup and its blowup*, Le Mathematiche **67** (2012), 75-80.

10. J. C. Rosales, *Numerical semigroups with multiplicity three and four*, Semigroup Forum **71**(2005), 323-331.

11. J. C. Rosales and P. A. Garcia-Sánchez, *Numerical Semigroups*, Springer, 2009.

12. J. C. Rosales, P. A. Garcia-Sánchez, J. I. Garcia-Garcia, and M. B. Branco, *Arf numerical semigroups*, J. of Alg. **276** (2004), 3-12.

13. M. Süer, H. İ. Karakas, and S. İlhan, *Arf numerical semigroups with multiplicity eight*, Int. J. Pure Appl. Sci. **9(2)** (2023), 393-401.

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