Some results on Wilf's conjecture

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Introduction

Let $S \subseteq \mathbb{N}$ be a numerical semigroup. Denote $S^* = S \setminus \{0\}$.

Notation

- $m = \min(S^*)$, the **multiplicity**
- $g = |\mathbb{N} \setminus S|$, the **genus**, i.e. number of gaps
- $D = S^* + S^*$, the **decomposable elements**
- $P = S^* \setminus D$, the **primitive elements**, i.e. minimal generators
- $L = S \cap [0, c-1]$, the **left part**, i.e. small elements
- $c =$ the **conductor**, i.e. least integer such that $c + \mathbb{N} \subseteq S$

Conjecture (Wilf 1978)

For any numerical semigroup S, $|P||L| \geq c$.

Main open cases in Wilf's conjecture:

- $3 < |P| < m/3$
- $c > 3m$
- $|L| > 12$
- $m > 19$
- $g > 66$

Some more notation

- $q = \lceil c/m \rceil$, the **depth**
- Thus $(q-1)m < c \leq qm$
- $A = S \setminus (m+S)$, the **Apéry set** with respect to m. Recall $|A| = m$.

In this talk, we focus on the case $c = qm$ and call it special.

Main results

Theorem [E, 2024+]

Let S be a numerical semigroup. If S is special and $|P| \geq m/4$ then S satisfies Wilf's conjecture.

See *Divsets, numerical semigroups and Wilf's conjecture*, https://hal.science/hal-04234167v4

Corollary [Delgado-E-Fromentin, 2024+]

Wilf's conjecture is true in the special case up to genus $g \leq 120$.

See *A verification of Wilf's conjecture up to genus* 100, https://arxiv.org/pdf/2310.07742.

▷ The proofs of both results are based on a clever trimming of the NS tree (Delgado 2019) and huge parallel computations.

▷ That paper also reveals the previously unknown numbers n_{73}, n_{74} and $n_{75} = 17,924,213,336,425$.

The numbers $W(S)$ and $W_0(S)$

Denote $W(S) = |P||L| - c$. Wilf's conjecture states: $W(S) \geq 0$.

Lemma

Assume $c = qm$. Let $L^\circ = L \setminus m\mathbb{N}$. Then $W(S) = |P||L^\circ| - |A \cap D|q$.

Proof

- ▷ $W(S) = |P||L| - c = |P||L| - qm$
- ▷ $m = |A| = |\{0\}| + |A \cap P| + |A \cap D| = |P| + |A \cap D|$
- ▷ $L \cap m\mathbb{N} = \{0, 1, \ldots, (q-1)\}m$, whence $|L \cap m\mathbb{N}| = q$
- ▷ Hence $|L| = |L \setminus m\mathbb{N}| + |L \cap m\mathbb{N}| = |L^\circ| + q$
- ▷ $W(S) = |P|(|L^\circ| + q) - mq = |P||L^\circ| - (m - |P|)q = |P||L^\circ| - |A \cap D|q$. $\square$

Upshot. Understanding $A \cap D$ is key for progress on Wilf's conjecture.

Denote $W_0(S) = |P \cap L||L^\circ| - |A \cap D|q = E(S)$. Then $W_0(S) \leq W(S)$.

Divsets

Definition

A *divset* is a finite set X of monomials in commuting variables $x_1, \ldots, x_n$ which is closed under taking divisors.

▷ That is, X is a finite *downset* or *order ideal* in $\mathcal{M} = \{x_1^{a_1} \cdots x_n^{a_n}\}$ under divisibility.

▷ **Example.** $X = \{x_1^3, x_1x_2^2, x_1^2, x_1x_2, x_2^2, x_1, x_2, 1\}$, the set of all divisors of $x_1^3, x_1x_2^2$. It is a divset. We say that X is spanned by $x_1^3, x_1x_2^2$ and we write

$$X = [x_1^3, x_1x_2^2].$$

▷ In a divset X, denote $X_d = \{u \in X \mid \deg(u) = d\}$. Thus $X_0 = \{1\}$ and $X_1 \subseteq \{x_1, \ldots, x_n\}$.

▷ Denote $D(X) = \{u \in X \mid \deg(u) \geq 2\}$, the set of decomposable monomials. Thus, $X = \{1\} \sqcup X_1 \sqcup D(X)$.

Divset models

Definition

Let $S \subseteq \mathbb{N}$ be a numerical semigroup of multiplicity m. Let $A \subset S$ be its Apéry set. A **divset model** of S is a divset X with a map $f\colon X \to S$ s.t.

- f is a morphism, i.e. $f(u_1 u_2) = f(u_1) + f(u_2)$ for all $u_1, u_2 \in X$
- f is injective
- $f(X) \subseteq A$
- $f(D(X)) = A \cap D$.

▷ The purpose of a divset model of S is to capture the fine structure of $A \cap D$, for a better understanding of Wilf's conjecture.

▷ The equality $f(X) = A$ is not required: elements of $A \cap P$ that divide no other Apéry element may be ignored.

▷ A single divset may represent many distinct numerical semigroups.

Example

▷ Among the more than 10^{13} numerical semigroups of genus $g \leq 60$, there are only *five* near-misses in Wilf's conjecture, i.e. with $W_0(S) < 0$.

▷ Recalling the notation $\langle a_1, \ldots, a_k \rangle_t = \langle a_1, \ldots, a_k \rangle \cup (t + \mathbb{N})$, here they are [E-Fromentin, 2019]:

$$\langle 14, 22, 23 \rangle_{56}$$
$$\langle 16, 25, 26 \rangle_{64}$$
$$\langle 17, 26, 28 \rangle_{68}$$
$$\langle 17, 27, 28 \rangle_{68}$$
$$\langle 18, 28, 29 \rangle_{72}$$

▷ Then all five near-misses are represented by the single divset

$$X = [x^3, x^2y, xy^2, y^3].$$

The graph of a divset

Let X be a divset. Let $X^* = X \setminus \{1\}$. The graph of X, denoted $G(X)$, is defined as follows.

- Its edges are all subsets $\{u, v\} \subseteq X^*$ such that $uv \in X^*$.
- Its vertices are all the extremities of the edges.

▷ Thus, a vertex in $G(X)$ is any $u \in X^*$ such that there exists $v \in X^*$ satisfying $uv \in X^*$.

▷ **Example.** Let $X = [x^2y, xz, w]$. The edges of $G(X)$ are

$$\{x, x\},\ \{x, y\},\ \{x, z\},\ \{x, xy\},\ \{y, x^2\}$$

and its vertices are x, y, z, xy, x^2.

The vertex-maximal matching number

Let G be a finite graph.

▷ A matching in G is a set of *pairwise disjoint* (i.e. *independent*) edges.

▷ The matching number of G is the largest size of a matching in G.

▷ The vertex-maximal matching number of G is the largest number of vertices contained in a matching of G.

Let X be a divset. We denote by $\mathrm{vm}(X)$ the vertex-maximal matching number of the graph $G(X)$.

▷ **Example**. Let $X = [x^3y]$. A large matching of $G(X)$:

$$\{x, x^2y\},\ \{y, x^3\},\ \{x^2, xy\}.$$

This is optimal. Thus $\mathrm{vm}(X) = 6$.

Links with $W(S)$

Let $S \subseteq \mathbb{N}$ be a special numerical semigroup, i.e. such that $c = qm$.
Recall that $L^\circ = L \setminus m\mathbb{N}$ and $|L| = |L^\circ| + q$.

Theorem

Let X be a divset model of S. Then $|L^\circ| \geq vm(X)q/2$. (See next slide)

Corollary

If $|P| \geq m/4$ and $\mathrm{vm}(X) \geq 6$, then $W(S) \geq 0$.

Proof.

$W(S) = |P||L| - c \geq (m/4)(6q/2 + q) - qm = 0.$ □

▷ So, to settle Wilf's conjecture in the special case with $|P| \geq m/4$, we need only consider divsets X such that $\mathrm{vm}(X) \leq 5$. A strong restriction.

Theorem

Let X be a divset model of S. Then $|L^\circ| \geq vm(X)q/2$.

Steps of proof

▷ $S = \bigsqcup_{a \in A} (a + m\mathbb{N})$

▷ $L = \bigsqcup_{a \in A} (a + m\mathbb{N}) \cap L$

▷ For $x \in S$, denote $\delta(x) = q - \lfloor x/m \rfloor$

▷ Then for all $a \in A$, $\delta(a) = |(a + m\mathbb{N}) \cap L|$

▷ Hence $|L| = \sum_{a \in A} \delta(a)$

▷ If $a, b \in S^*$ and $a + b \in A \cap D$, then $a, b \in A \cap L^\circ$ and $\delta(a) + \delta(b) \geq q$

▷ Each such pair $\{a, b\} \subseteq A \cap L^\circ$ corresponds to an edge in $G(X)$

▷ Hence each vertex in a vertex-maximal matching of $G(X)$ contributes to at least $q/2$ in average to $\mathrm{vm}(X)$. □

Proposition

If $\mathrm{vm}(X) \leq 5$, X contains no monomials of the form x_1^7, $x_1^3 x_2$ or $x_1 x_2 x_3$.

Proof.

Each case implies $\mathrm{vm}(X) \geq 6$. Indeed:

- $x_1^7 \in X \Rightarrow$ matching $\{x_1, x_1^6\}, \{x_1^2, x_1^5\}, \{x_1^3, x_1^4\}$ on 6 vertices.
- $x_1^3 x_2 \in X \Rightarrow$ matching $\{x_1, x_1^2 x_2\}, \{x_2, x_1^3\}, \{x_1^2, x_1 x_2\}$.
- $x_1 x_2 x_3 \in X \Rightarrow$ matching $\{x_1, x_2 x_3\}, \{x_2, x_1 x_3\}, \{x_3, x_1 x_2\}$.

□

The degree of a divset

Let X be a divset. Its *degree* is $\deg(X) = \max\{\deg(u) \mid u \in X\}$.

Proposition

Let X be a divset of degree ≤ 2. Then all special numerical semigroups S modelled by X satisfy $W(S) \geq 0$.

Proof.

By graph theory, counting edges in a bipartite graph with given matching number. □

Question

For a divset X in n variables, of degree d and number $v = \mathrm{vm}(X)$, what is the largest possible cardinality of $D(X)$?

Since $|D(X)| = |A \cap D|$, good answers to this question might be useful for progress on Wilf's conjecture.

Tame divsets

A divset X on n variables is **tame** if $(n+1)\,\mathrm{vm}(X) \geq 2|D(X)|$, **wild** otherwise.

Proposition

All special numerical semigroups modelled by a tame divset satisfy Wilf's conjecture.

Proof.

We have $|P| \geq n+1$, $|L^\circ| \geq \mathrm{vm}(X)q/2$ and $|A \cap D| = |D(X)|$. Hence

$$\begin{aligned} W(S) &= |P||L^\circ| - |A \cap D|q \\ &\geq ((n+1)\,\mathrm{vm}(X) - 2|D(X)|)q/2 \\ &\geq 0. \quad \square \end{aligned}$$

Relevant divsets

It remains to consider divsets X such that $\mathrm{vm}(X) \leq 5$, $\deg(X) \geq 3$ and $n \geq 3$. Those maximizing $|D(X)|$ relative to $\deg(X)$ are as follows:

- If $\mathrm{vm}(X) = 2$, then $X = [x_1^3, x_1x_2, \dots, x_1x_n]$.
- If $\mathrm{vm}(X) = 3$, then
  - $X = [x_1^3, x_1x_2, \dots, x_1x_n, x_2^2]$
  - $X = [x_1^4, x_1x_2, \dots, x_1x_n]$
- If $\mathrm{vm}(X) = 4$, then
  - $X = [x_1^3, x_1x_2, \dots, x_1x_n, x_2^2, x_2x_3, \dots, x_2x_n]$
  - $X = [x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3, \dots, x_1x_n, x_2x_3, \dots, x_2x_n]$
  - $X = [x_1^3, x_1^2x_2, \dots, x_1^2x_n]$
  - $X = [x_1^4, x_1^2x_2, \dots, x_1^2x_n, x_2^2]$
  - $X = [x_1^5, x_1^2x_2, \dots, x_1^2x_n]$
- And finally, a similar short list if $\mathrm{vm}(X) = 5$. (See preprint on HAL.)

All these divsets turn out to be tame – by direct computation of $|D(X)|$ and $\mathrm{vm}(X)$ – except for

$$X = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1 x_3, \dots, x_1 x_n, x_2 x_3, \dots, x_2 x_n].$$

For it, there is an ad-hoc proof that the special numerical semigroups modelled by it satisfy Wilf's conjecture.

Conclusion

All special num. semig. S such that $|P| \geq m/4$ satisfy $W(S) \geq 0$.

Corollary [Delgado-E-Fromentin, 2024+]

All special num. semig. S of genus $g \leq 120$ satisfy Wilf's conjecture.

Proof.

By a clever trimming of the NS tree (Delgado 2019) and huge parallel computations. □

Perspectives

▷ To extend this result to the case $|P| \geq m/5$, one needs to further analyze all divsets X satisfying $\mathrm{vm}(X) \in \{6,7\}$.

▷ There will be more wild cases requiring ad-hoc proofs.

▷ Understanding $(n+1)\mathrm{vm}(X)/2 - |D(X)|$ for divsets X in n variables is of independent interest.

Thank you for your attention.

Gracias por su atención.