Some asymptotic properties of shifted numerical semigroups

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International Meeting on Numerical Semigroups

Jerez de la Frontera, 12th July 2024

Joint work (in progress) with Dumitru Stamate

Shifted numerical semigroups

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We are interested in the properties of the ***shifted semigroups*** M_{n+m}.

Theorem (Vu, 2014) *The Betti numbers of the defining ideal of* M_{n+m} *are eventually periodic in* m *with period* $n_k - n$.

This result was conjectured by Herzog and Srinivasan.

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Example Consider $M_{53} = \langle 53, 55, 59, 60 \rangle$. We have $n = 53$, $r_1 = 2$, $r_2 = 6$, $r_3 = 7$, $d = 1$, therefore $S = \langle 2, 6, 7 \rangle = \langle 2, 7 \rangle$. In this case $17 = 2 \times 2 + 6 + 7$ and then $\mathbf{m}(17) = 4$.

Apéry set

Theorem (O'Neill, Pelayo, 2018) *If $n > r_k^2$, then*

$$\mathrm{Ap}(M_n, n) = \{i + \mathbf{m}(i)n \mid i \in \mathrm{Ap}(S, dn)\}.$$

Moreover, for each $i \in \mathrm{Ap}(S, dn)$, all the factorizations of $i + \mathbf{m}(i)n$ in M_n have length $\mathbf{m}(i)$.

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If $dn > r_k^2$, then $\mathrm{Ap}(S, dn) = \{i_0, \dots, i_{n-1}\}$, where

$$i_j = \begin{cases} dj & \text{if } dj \in S \\ dj + dn & \text{if } dj \notin S \end{cases}$$

Note that $|\mathrm{Ap}(S, dn)| = n$.

A bijection

Let P_n denote the set

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$$i \mapsto \begin{cases} i & \text{if } i < dn - r_k \\ i + dr_k & \text{if } i \geq dn - r_k \end{cases}$$

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Example Let $M_{53} = \langle 53, 55, 59, 60 \rangle$ and $M_{53+7} = M_{60} = \langle 60, 62, 66, 67 \rangle$:

$$\mathrm{PF}(M_{53}) = \{176, 421, 425, 482\} \qquad P_{53} = \{17, 50, 54, 58\}$$

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Pseudo-Frobenius numbers

Corollary *Let* $n \gg 0$. *If* $f \in \mathrm{PF}(M_n)$, $f \equiv i \mod n$ *with* $i \in \mathrm{Ap}(S, dn)$*:*

$$\varphi_n(f) = \begin{cases} f + (\mathbf{m}(i) - 1)r_k & \textit{if } i < dn - r_k \\ f + (\mathbf{m}(i) + 2d - 1)r_k + dn & \textit{if } i \geq dn - r_k \end{cases}$$

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$$\varphi_{53}(176) = 176 + (4 - 1) \times 7 = 197$$

$$\varphi_{53}(421) = 421 + (8 + 1) \times 7 + 53 = 537$$

$$\varphi_{53}(425) = 425 + (8 + 1) \times 7 + 53 = 541$$

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Since $i > dn - r_k$ is equivalent to $i + dr_k > d(n + r_k) - r_k$, it follows

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Given $f \in \mathrm{PF}(M_n)$, we denote by $\varphi_n^\lambda(f)$ the image of f via the map

$$\varphi_{n+(\lambda-1)r_k} \circ \varphi_{n+(\lambda-2)r_k} \circ \cdots \circ \varphi_{n+r_k} \circ \varphi_n.$$

Hence, $\varphi_n^\lambda(f) \in \mathrm{PF}(M_{n+\lambda r_k})$.

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Corollary *Let* $n \gg 0$. *Let* $f \in \mathrm{PF}(M_n)$ *be such that* $f \equiv i \mod dn$ *with* $i \in \mathrm{Ap}(S, dn)$. *Then*

$$\varphi_n^\lambda(f) = \begin{cases} f + (\mathbf{m}(i) - 1)\lambda r_k & \textit{if } i < n - r_k, \\ f + (\mathbf{m}(i) + (\lambda+1)d - 1)\lambda r_k + \lambda dn & \textit{if } i \geq n - r_k. \end{cases}$$

Frobenius number

Let $n \gg 0$. Let $f_1, f_2, \dots, f_\alpha \in \mathrm{PF}(M_n)$ corresponding to elements in P'_n and let $g_1, \dots, g_\beta \in \mathrm{PF}(M_n)$ corresponding to elements in P''_n.

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For n big enough we have $f_1 < \cdots < f_\alpha < g_1 < \cdots < g_\beta$ and

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Proposition *Let* $n \gg 0$ *and* $\mathrm{F}(M_n) \equiv i \mod dn$ *with* $i \in \mathrm{Ap}(S, dn)$. *Then* $i \geq dn - r_k$ *and for every* $\lambda \in \mathbb{N}$ *holds*

$$\mathrm{F}(M_{n+\lambda r_k}) = \varphi_n^\lambda(\mathrm{F}(M_n)) = \mathrm{F}(M_n) + (\mathbf{m}(i) + (\lambda+1)d - 1)\lambda r_k + \lambda dn$$

Nearly Gorenstein numerical semigroups

Let H be a numerical semigroup and $K(H) = \{x \in \mathbb{N} | \operatorname{F}(H) - x \notin H\}$.
The *trace ideal* of $K(H)$ is $\operatorname{tr}(H) = K(H) + (H - K(H))$.

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Definition Let H be minimally generated by $h_1, \ldots, h_k$. We say that $(f_1, \ldots, f_k) \in \mathrm{PF}(H)^k$ is an ***NG-vector*** for H, if for all $f \in \mathrm{PF}(H)$ and i

$$h_i + f_i - f \in H.$$

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Proposition (Moscariello, S., 2021) *H is nearly Gorenstein if and only if there exists an NG-vector for H.*

Nearly Gorensteinness is periodic

Theorem *If* $n \gg 0$ *and* M_n *is nearly Gorenstein, then* $M_{n+\lambda r_k}$ *is nearly Gorenstein for every* $\lambda \in \mathbb{N}$.
Moreover, if $(f_0, \ldots, f_k)$ *is an NG-vector for* M_n*, then* $(\varphi_n^\lambda(f_0), \ldots, \varphi_n^\lambda(f_k))$ *is an NG-vector for* $M_{n+\lambda r_k}$.

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Example Let $M_{30} = \langle 30, 32, 33, 35 \rangle$. Then, $\mathrm{PF}(M_{30}) = \{209, 211\}$ and M_{30} is nearly Gorenstein with NG-vector $(211, 209, 211, 209)$. It follows that $M_{30+5\lambda}$ is nearly Gorenstein with NG-vector

$$(211 + 65\lambda + 5\lambda^2, 209 + 65\lambda + 5\lambda^2, 211 + 65\lambda + 5\lambda^2, 209 + 65\lambda + 5\lambda^2)$$

for all $\lambda \in \mathbb{N}$.

Almost symmetric numerical semigroups

Theorem (Nari, 2013) *Let* $H = \langle h_1, \ldots, h_k \rangle$ *be a numerical semigroup. It is* ***almost symmetric*** *if and only if* $\mathrm{F}(H) - f \in \mathrm{PF}(H)$ *for every* $f \in \mathrm{PF}(H) \setminus \{\mathrm{F}(H)\}$*, that is*

$$\mathrm{F}(H) - f + h_j \in H$$

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Corollary *Let $n \gg 0$. If M_n is almost symmetric, then $M_{n+\lambda r_k}$ is almost symmetric for every $\lambda \in \mathbb{N}$.*

Almost symmetric semigroups with even type

Herzog and Watanabe (2019) proved that, when M_n has 4 generators, there are only finitely many m for which M_{n+m} is almost symmetric of type 2.

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However, this is not true when the type is odd.

Example (Numata) Let $M_{10} = \langle 10, 11, 13, 14 \rangle$. Then

$$M_{10+4\lambda} = \langle 10 + 4\lambda, 11 + 4\lambda, 13 + 4\lambda, 14 + 4\lambda \rangle$$

is almost symmetric of type 3 for every $\lambda \in \mathbb{N}$.

Numerical semigroups having canonical reduction

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Corollary *If $n \gg 0$ and M_n has a canonical reduction, then $M_{n+\lambda r_k}$ has a canonical reduction for every $\lambda \in \mathbb{N}$.*

Residue

The ***residue*** of H is defined as $\mathrm{res}(H) = |H \setminus \mathrm{tr}(H)|$.

$$\mathrm{res}(H) = 0 \iff H \text{ is symmetric}$$
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One may expect that the residue is periodic, but this is not true.

Example Let $M_{46+11\lambda} = \langle 46 + 11\lambda, 48 + 11\lambda, 52 + 11\lambda, 57 + 11\lambda \rangle$.
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Question *Is* $\mathrm{res}(M_{n+\lambda r_k})$ *linear in* λ*?*

Reduced type

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Proposition *Assume* $n \gg 0$. *Then,* $s(\mathbb{k}[[M_{n+r_k}]]) = s(\mathbb{k}[[M_n]])$.

Thank you for your attention!