The far-flung Gorenstein property for numerical semigroups

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Far-flung Gorenstein rings

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module ω_R.

Then R is a Gorenstein ring $\iff \operatorname{inj\,dim}_R R < \infty \iff \omega_R \cong R$.

Aim: stratify the space between Cohen-Macaulay rings and Gorenstein rings

Classes of rings: almost Gorenstein, nearly Gorenstein, quasi-Gorenstein, level, pseudo Gorenstein, generalized Gorenstein etc., some motivated by the study of numerical semigroups.

Tools: the size of Hilbert coefficients, trace ideals, the entries in the Betti table, etc.

Far-flung Gorenstein rings

$(R, \mathfrak{m})$ is a Cohen-Macaulay local ring, $\dim R = 1$ with canonical module ω_R.

- the canonical trace ideal is

$$\operatorname{tr}(\omega_R) = \sum_{\varphi:\omega_R \to R} \operatorname{Im} \varphi \subseteq R,$$

- $\overline{R}$ = the integral closure of R
- $Q(R)$ = the total ring of fractions of R
- The conductor of $R \subseteq \overline{R}$ is $R :_{Q(R)} \overline{R} = \{x \in Q(R) : x\overline{R} \subseteq R\}$.
- Fact: $\operatorname{tr}(\omega_R) \supseteq R :_{Q(R)} \overline{R}$.

Definition

The ring R is called *far-flung Gorenstein* (FFG) if

$$\operatorname{tr}(\omega_R) = R :_{Q(R)} \overline{R}.$$

Far-flung Gorenstein rings–characterizations

$(R, \mathfrak{m})$ is a Cohen-Macaulay local ring, $\dim R = 1$ with canonical module ω_R.
Technical conditions:

1. There exists an R-submodule $C \subseteq Q(R)$ such that $R \subseteq C \subseteq \overline{R}$ and $C \cong \omega_R$.
2. $\overline{R}$ is a finitely generated R-module
3. $\overline{R}$ is a local ring with maximal ideal $\mathfrak{n}$
4. there exists a non-zerodivisor $a \in \mathfrak{m}$ such that (a) is a reduction of $\mathfrak{m}$.

Theorem

Assume the technical conditions above hold. Then the following are equivalent:

1. *R is a far-flung Gorenstein ring;*
2. $\operatorname{tr}_R(\omega_R) \cong \overline{R}$;
3. $C^2 = \overline{R}$.

FFG-numerical semigroups

K field, H numerical semigroup
Then $K[|H|]$ is a 1-dimensional Cohen-Macaulay domain.
Aim: describe the far-flung Gorenstein property for $R = K[|H|]$

- $e(H)$ = multiplicity of H
- $F(H)$ = Frobenius number of H
- the pseudo-Frobenius numbers

$PF(H) = \{x \in \mathbb{Z} \setminus H : x + h \in H \text{ for all } 0 \neq h \in H\}$ determine

- the canonical module $\omega_R = (t^{-\alpha} : \alpha \in PF(H))R$.
- the type of H is $r(H) = |PF(H)|$, equals the Cohen-Macaulay type of $K[|H|]$.
- use a copy of ω_R,

$$C = (t^{F(H)-\alpha} : \alpha \in PF(H))R$$

Then $R \subseteq C \subseteq \overline{R} = K[|t|]$.

Theorem

The ring $K[|H|]$ is far-flung Gorenstein if and only if $C^2 = K[|H|]$.

FFG-semigroups — characterizations

$$C = (t^{F(H)-\alpha} : \alpha \in PF(H))K[|H|]$$

Theorem

Let H be a numerical semigroup and K any field. The following statements are equivalent:

1. *the ring $K[|H|]$ is far-flung Gorenstein;*
2. $\{0, \ldots, e(H)-1\} \subseteq \{2F(H)-\alpha-\beta : \alpha, \beta \in PF(H)\}$;
3. $\{2F(H)-e(H)+1, \ldots, 2F(H)\} \subseteq \{\alpha+\beta : \alpha, \beta \in PF(H)\}$.

Consequently, the far-flung property of $K[|H|]$ depends on H and not on the field K.

Example

$H = \langle 7, 8, 11, 17, 20\rangle$ is far-flung Gorenstein.
$PF(H) = \{9, 10, 12, 13\}$, hence $C = (1, t, t^3, t^4)K[|H|]$.
$C^2 = (1, t, t^2, t^3, t^4, t^5, t^6, t^7, t^8)K[|H|] = K[|t|]$.

Corollary

Let H be a numerical semigroup minimally generated by $a_1 < \cdots < a_v$ which is of minimal multiplicity, i.e. $v = a_1$. Then $K[|H|]$ is a far-flung Gorenstein ring if and only if

$$\{2a_v - a_1 + 1, \dots, 2a_v\} \subseteq \{a_i + a_j : 2 \leq i, j \leq v\}.$$

We thus obtain an infinite family of far-flung numerical semigroups.

Corollary

Let $a \geq 3$ and d be coprime nonnegative integers and $H = \langle a, a + d, \dots, a + (a-1)d \rangle$. Then $R = K[|H|]$ is a far-flung Gorenstein ring if and only if $d = 1$.

FFG semigroups-the multiplicity

Theorem

If R is a far-flung Gorenstein ring with Cohen-Macaulay type $r(R)$ and multiplicity $e(R)$, then

$$r(R) + 1 \leq e(R) \leq \binom{r(R)+1}{2}.$$

The lower bound characterizes in fact when the ring R has minimal multiplicity, but the upper bound may not always be sharp.

This gives automatically the following . . .

FFG semigroups

Theorem

If H is a far-flung Gorenstein numerical semigroup with type $r(H)$ and multiplicity $e(H)$, then

$$r(H)+1 \leq e(H) \leq \binom{r(H)+1}{2}.$$

• If $\operatorname{emb\,dim}(H) = 3$ and H is FFG then $r(H) = 2$ and $3 \leq e(H) \leq 3$, i.e. $e(H) = 3$ and H has minimal multipilicity.

Proposition

If $\operatorname{emb\,dim}(H) = 3$*, then*

$$H \text{ is far-flung Gorenstein} \iff H = \langle 3, 3n+1, 3n+2 \rangle, n \geq 1.$$

• Can we get a sharper (upper) bound for numerical semigroup rings ?

The Rohrbach problem

Definition

Let $A \subseteq \mathbb{N}$ with r elements. Let $n(A)$ denote the integer such that the sum-set

$$A + A = \{a + b : a, b \in A\}$$

contains the integers $0, 1, \ldots, n(A) - 1$ but not $n(A)$.
If $0 \notin A$ then let $n(A) = -1$.

- $A = \{0, 1, 2, 3\} \Rightarrow A + A = \{0, 1, 2, 3, 4, 5, 6\} \Rightarrow n(A) = 7$
- $A = \{0, 1, 3, 4\} \Rightarrow A + A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \Rightarrow n(A) = 9$.

Problem

For $r > 0$ the Rohrbach problem asks to find the integer

$$\overline{n}(r) = \max\{n(A) : |A| = r\}.$$

The solution $\overline{n}(r)$ of the Rohrbach problem is known for $r \leq 25$

r	1	2	3	4	5	6	7	8	9
$\overline{n}(r)$	1	3	5	9	13	17	21	27	33
r	10	11	12	13	14	15	16	17	18
$\overline{n}(r)$	41	47	55	65	73	81	93	105	117
r	19	20	21	22	23	24	25	$\cdots$	$\cdots$
$\overline{n}(r)$	129	141	153	165	181	197	213	$\cdots$	$\cdots$

Proposition

If H is a far-flung Gorenstein numerical semigroup, then

$$e(H) \leq \overline{n}(r), \ \textit{where } r = r(H), \textit{ the type of } H.$$

Note that already when $r = 3$ one has $5 = \overline{n}(3) < \binom{3+1}{2} = 6$, hence the solution to the Rohrbach problem gives a better upper bound than the one known before $\binom{r(H)+1}{2}$.

FFG-small type

When the type of H is small, one gets a full classification.

Proposition

Assume the type $r(H) = 2$. Then $K[|H|]$ is far flung Gorenstein ring if and only if $H = \langle 3, 3n+1, 3n+2 \rangle$ for some integer $n > 0$.

Theorem

Assume $r(H) = 3$ and H is not of minimal multiplicity. Then $K[|H|]$ is a far-flung Gorenstein ring if and only if H is in one of the following families of semigroups

1. $H = \langle 5, 5m+4, 10m+6, 10m+7 \rangle$, *where* $m \geq 1$;
2. $H = \langle 5, 5m+1, 10m+3, 10m+4 \rangle$, *where* $m \geq 1$;
3. $H = \langle 5, 5m+2, 10m+1, 10m+3 \rangle$, *where* $m \geq 1$;
4. $H = \langle 5, 5m+3, 10m+4, 10m+7 \rangle$, *where* $m \geq 1$.

• T. Grigorescu recently classified the FFG numerical semigroups of type 4.

Question

Do exist far-flung Gorenstein numerical semigroups H of type r where $e(H) = \overline{n}(r)$, for any r?

Thank you!

Muchas gracias por su atención!