A linear variant of nearly Gorensteinness and projective monomial curves

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Projective monomial curves

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of non-negative integers.

Let $\mathbf{a} = a_1, a_2, \cdots, a_n \in \mathbb{Z}_{>0}$ with $\gcd(a_1. \cdots, a_n) = 1$.

We may assume that $0 < a_{i_1} < a_{i_2} < \cdots < a_{i_n}$, then

$$S_{\mathbf{a}} := \mathbb{N}(0, a_{i_n}) + \mathbb{N}(a_{i_1}, a_{i_n} - a_{i_1}) + \mathbb{N}(a_{i_2}, a_{i_n} - a_{i_2}) + \cdots + \mathbb{N}(a_{i_{n-1}}, a_{i_n} - a_{i_{n-1}}) + \mathbb{N}(a_{i_n}, 0)$$

We call $S_{\mathbf{a}}$ the *projective monomial curve* defined by $\mathbf{a}$.

Ex

Set $S = \mathbb{N}(0, 25) + \mathbb{N}(7, 25 - 7) + \mathbb{N}(9, 25 - 9) + \mathbb{N}(16, 25 - 16) + \mathbb{N}(25, 0)$.
Then S is projective monomial curve defined by $7, 9, 16, 25$.

We can study $S_{\mathbf{a}}$ by using the techniques of numerical semigroups!

Affine semigroup rings

An *affine semigroup* S is a finitely generated sub-semigroup of $\mathbb{Z}^d$.

$$\Bbbk[S] := \Bbbk[\mathbf{x}^{\mathbf{p}} : \mathbf{p} \in S]$$

is called *affine semigroup rings* of S. $\mathbf{x}^{\mathbf{p}} \cdot \mathbf{x}^{\mathbf{q}} = \mathbf{x}^{\mathbf{p}+\mathbf{q}} \ \forall\, \mathbf{p}, \mathbf{q} \in S.$

Def

Let $\mathbf{a} = a_1, a_2, \cdots, a_n \in \mathbb{Z}_{>0}$ with $\gcd(a_1. \cdots, a_n) = 1$ and let $\Bbbk$ be a field.

We also call $\Bbbk[S_{\mathbf{a}}]$ the *projective monomial curve* defined by $\mathbf{a}$.

Semi-standard graded rings and its h-vector

$R = \bigoplus_{i \geq 0} R_i$: a positively graded ring with $R_0 = \Bbbk$ a field.

Def

- R is *standard graded* $\overset{\text{def}}{\Leftrightarrow}$ $R = \Bbbk[R_1]$.
- R is *semi-standard graded* $\overset{\text{def}}{\Leftrightarrow}$ R is finitely generated as $\Bbbk[R_1]$-module.

Notice that the projective monomial curve

$$\Bbbk[S_{\mathbf{a}}] \cong \Bbbk[t^{a_{i_n}}, s^{a_{i_1}} t^{a_{i_n} - a_{i_1}}, s^{a_{i_2}} t^{a_{i_n} - a_{i_2}}, \cdots, s^{a_{i_{n-1}}} t^{a_{i_n} - a_{i_{n-1}}}, s^{a_{i_n}}]$$

is a standard graded ring with

$$\deg t^{a_{i_n}} = \deg s^{a_{i_1}} t^{a_{i_n} - a_{i_1}} = \deg s^{a_{i_2}} t^{a_{i_n} - a_{i_2}} = \cdots = \deg s^{a_{i_{n-1}}} t^{a_{i_n} - a_{i_{n-1}}} = \deg s^{a_{i_n}} = 1.$$

Def

If R is a semi-standard graded ring, then

$$\mathrm{Hilb}(R,t) := \sum_{i\geq 0} (\dim_{\Bbbk} R_i) t^i = \frac{h_0 + h_1 t + \cdots + h_{s(R)} t^{s(R)}}{(1-t)^{\dim R}}$$

where $h_i \in \mathbb{Z}$ and $h_{s(R)} \neq 0$. $h(R) = (h_0, h_1, \cdots, h_{s(R)})$ is called *h-vector* of R.

$s(R)$ is called *socle degree* of R.

Nearly Gorenstein

Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be a positively graded ring with $R_0 = \Bbbk$. Assume that R is CM.

$\mathfrak{m} := \oplus_{i>0} R_i$. ω_R : canonical module of R. $\mathrm{tr}_R(\omega_R) := \sum_{\phi\in\mathrm{Hom}(M,R)} \phi(M) \subset R$.

Def and **Rem** (Herzog–Hibi–Stamate (2019))

- R is *nearly Gorenstein* $\Leftrightarrow \mathrm{tr}_R(\omega_R) \supset \mathfrak{m}$.
- R is Gorenstein on $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\} \Leftrightarrow \sqrt{\mathrm{tr}_R(\omega_R)} \supset \mathfrak{m}$.

In particular, R is nearly Gorenstein $\Rightarrow$ R is Gorenstein on $\mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$.

Nearly Gorenstein projective monomial curves with $n \leq 4$

Note that $\Bbbk[S_{a_1,a_2}]$ is always Gorenstein.

Fact (M.)

Assume that $\Bbbk[S_{\mathbf{a}}] = \Bbbk[S_{a_1,a_2,\cdots,a_n}]$ is Cohen–Macaulay. The following is true:

(1) If $n = 3$, then $\Bbbk[S_{\mathbf{a}}]$ is non-Gorenstein and nearly Gorenstein
$\iff \mathbf{a} = k, k+1, 2k+1$ $(k \geq 1)$.

(2) If $n = 4$, $\Bbbk[S_{\mathbf{a}}]$ is non-Gorenstein and nearly Gorenstein
$\iff \mathbf{a} = 1, 2, 3, 4$ or $S_{\mathbf{a}} \cong S_{2k-1,2k+1,4k,6k+1}$ $(k \geq 1)$.

To prove it, the following fact is a key.

Fact (M.) Assume that $R = \Bbbk[S_{a_1,\cdots,a_n}]$ is Cohen–Macaulay.
If R is non-Gorenstein and $\mathrm{tr}(\omega_R) \supset (s^{a_{i_n}}, t^{a_{i_n}})$, then $h_{s(R)} \geq 2$.

Projective monomial curves with $\mathrm{tr}(\omega_R) \supset (s^{a_{i_n}}, t^{a_{i_n}})$

Note

Assume $\Bbbk[S_\mathbf{a}] = \Bbbk[S_{a_1,a_2,a_3,a_4}]$ is CM but not Gorenstein. The following is true:
$\Bbbk[S_\mathbf{a}]$ is nearly Gor $\Leftrightarrow$ $\mathbf{a} = 1, 2, 3, 4$ or $S_\mathbf{a} \cong S_{2k-1,2k+1,4k,6k+1}$ $(k \geq 1)$.

In this case $(n = 4)$, if we consider the condition $\mathrm{tr}(\omega_R) \supset (s^{a_{i_4}}, t^{a_{i_4}})$ instead of the nearly Gorenstein property, we can obtain many more examples.

Ex

Take any $k \in \mathbb{Z}_{>1}$. Set $\mathbf{a} = 4k+2, 4k+6, 6k+5, 10k+11$ or $\mathbf{a} = 1, k, 2k, 3k$.
Then $R = \mathbb{Q}[S_\mathbf{a}]$ is CM. Moreover, R is not nearly Gor but $\mathrm{tr}(\omega_R) \supset (s^{a_{i_n}}, t^{a_{i_n}})$.

Property $\mathrm{tr}(\omega_R) \supset (s^{a_{i_n}}, t^{a_{i_n}})R$ can be understood as a property of more general affine semigroup rings as follows:

Let S be an affine semigroup. We assume that $S \cap (-S) = \{0\}$ and $S \subset \mathbb{N}^d$.

- $R := \Bbbk[S]$: CM semi-standard graded affine semigroup ring.
- $G_S \subset \mathbb{N}^d$: minimal generating system of S.
- $E_S := \{\mathbf{e} \in G_S : \mathbf{x}^{\mathbf{e}}$ *is corresponding to 1-dim face of* $\mathbb{R}_{\geq 0}S\}$.

Fact (M.)

If R is non-Gorenstein and $\mathrm{tr}(\omega_R) \supset (\mathbf{x}^{\mathbf{e}} : \mathbf{e} \in E_S)$, then $h_{s(R)} \geq 2$.

R is nearly Gorenstein $\Rightarrow \mathrm{tr}(\omega_R) \supset (\mathbf{x}^{\mathbf{e}} : \mathbf{e} \in E_S) \Rightarrow R$ is Gorenstein on $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$

Ehrhart rings

- $\Bbbk$: field.
- $P \subset \mathbb{R}^d$: lattice polytope

Def

Ehrhart ring of P is defined as $A(P) := \Bbbk[\mathbf{t}^\alpha t_{d+1}^n : n \in \mathbb{Z}_{>0},\ \alpha \in nP \cap \mathbb{Z}^d]$,
where $\mathbf{t}^\alpha := t_1^{\alpha_1} \cdots t_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$.

- $A(P)$ is CM semi-standard graded $\Bbbk$-algebra with $\deg(\mathbf{t}^\alpha t_{d+1}^n) = n$.

- For polytope $P, Q \subset \mathbb{R}^d$, $P + Q := \{p + q \in \mathbb{R}^d : p \in P, q \in Q\}$.
- $P \subset \mathbb{R}^d$ be a lattice polytope.
- Its *floor* $\lfloor P \rfloor$ and remainder $\{P\}$ defined by [Hall–Kölbl–Matsushita–M.].
- Set $[P] := \lfloor a_P P \rfloor$, where $a_P := \min\{k \in \mathbb{Z}_{>0} : \mathrm{int}(kP) \cap \mathbb{Z}^d \neq \emptyset\}$.

Fact (Hall–Kölbl–Matsushita–M.) $R = A(P)$.

R is nearly Gorenstein $\Rightarrow$ $P = [P] + \{P\}$ $\Rightarrow$ R is Gorenstein on $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$.

$$\textit{Nearly Gorenstein} \Rightarrow \begin{matrix} \mathrm{tr}(\omega_R) \supset (\mathbf{x}^{\mathbf{e}} : \mathbf{e} \in E_{G_S})R \\ P = [P] + \{P\} \end{matrix} \Rightarrow \textit{Gorenstein on } \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$$

Question

- Do the two conditions in the middle above have any relation?
- Can above condition be captured as a property of general rings?

Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be a Cohen–Macaulay semi-standard graded ring.

Def

R satisfies $(\natural) \Leftrightarrow \sqrt{\mathrm{tr}_R(\omega_R)_1 R} \supset \mathfrak{m}$.

Rem

R is nearly Gorenstein $\Rightarrow$ R satisfies $(\natural)$ $\Rightarrow$ R is Gorenstein on $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$.

Thm (M.) Let $\Bbbk[S]$ be a CM semi-standard graded affine semigroup ring, then

$\mathrm{tr}(\omega_{\Bbbk[S]}) \supset (\mathbf{x}^{\mathbf{e}} : \mathbf{e} \in E_S)\Bbbk[S] \Rightarrow \Bbbk[S]$ satisfies $(\natural)$. $\Leftarrow$ is also true if $|\Bbbk| = \infty$.

$\Updownarrow \leftarrow P$ is a lattice polytope. Set $C_P = \mathbb{R}_{\geq 0}(P \times \{1\})$ and $S = C_P \cap \mathbb{Z}^{d+1}$

$P = [P] + \{P\}$

Veronese subrings of rings satisfying condition ($\natural$)

Def

Let R be a positively graded ring and let $n > 0$ be an integer.

$R_{(n)} := \oplus_{i \geq 0} R_{in}$ is called the n-th Veronese subring of R.

Fact A (Herzog–Hibi–Stamate (2019))

Let R be a standard graded ring with $\dim R > 0$.

If R is Gorenstein, then $R_{(n)}$ is nearly Gorenstein $\forall n > 0$.

Fact B (Hall–Kölbl–Matsushita–M.)

Let P be a lattice polytope.

If $P = [P] + \{P\}$, then $A(nP) \cong A(P)_{(n)}$ is nearly Gorenstein for $n \gg 0$.

Q.

Can we generalize above results by using ($\natural$)?

$R = \bigoplus_{i \geq 0} R_i$: a CM semi-standard graded ring with $\dim R > 0$.

Thm A (M.)

(1) If R satisfies ($\natural$), then so does $R_{(n)}$ for any $n > 0$;

(2) If R is standard graded and satisfies ($\natural$), then the following is true:

(X) $R_{(n)}$ is nearly Gorenstein for any $n > s(R/\mathrm{tr}(\omega_R)_1 R)$;

(Y) In particular, if R is nearly Gorenstein, then so is $R_{(n)}$ for any $n > 0$.

Thm B (M.) Assume that R is a semi-standard graded domain.

If R satisfies ($\natural$), then $\exists$ a integer $k_R \geq 0$ s.t. $R_{(n)}$ is nearly Gorenstein $\forall n > k_R$.

$R = \bigoplus_{i \geq 0} R_i$: a CM semi-standard graded ring with $\dim R > 0$.

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Fact A (Herzog–Hibi–Stamate (2019))

Let R be a standard graded ring with $\dim R > 0$.

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$R = \bigoplus_{i \geq 0} R_i$: a CM semi-standard graded ring with $\dim R > 0$.

Thm B (M.) Assume that R is a semi-standard graded domain.
If R satisfies ($\natural$), then $\exists$ a integer $k_R \geq 0$ s.t. $R_{(n)}$ is nearly Gorenstein $\forall n > k_R$.

Recall that the following.

Thm (M.) Let P be a lattice polytope. If $P = [P] + \{P\}$, then R satisfies ($\natural$).

Thus, the following follows from above two Thorems.

Fact B (Hall–Kölbl–Matsushita–M.)
Let P be a lattice polytope.
If $P = [P] + \{P\}$, then $A(nP) \cong A(P)_{(n)}$ is nearly Gorenstein for $n \gg 0$.

Thm A (M.)

(1) If R satisfies ($\natural$), then so does $R_{(n)}$ for any $n > 0$;

(2) If R is standard graded and satisfies ($\natural$), then the following is true:

   (X) $R_{(n)}$ is nearly Gorenstein for any $n > s(R/\mathrm{tr}(\omega_R)_1 R)$;

   (Y) In particular, if R is nearly Gorenstein, then so is $R_{(n)}$ for any $n > 0$.

Thm B (M.) Assume that R is a semi-standard graded domain.

If R satisfies ($\natural$), then $\exists$ a integer $k_R \geq 0$ s.t. $R_{(n)}$ is nearly Gorenstein $\forall n > k_R$.

Rem 1

Thm A (X) and Thm B fails if we replace ($\natural$) with Gorenstein on $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$.

Indeed, there exists a Cohen–Macaulay standard graded affine semigroup ring R which is Gorenstein on $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ but $R_{(k)}$ does not satisfy ($\natural$) for any $k > 0$.

Thm A (M.)

(1) If R satisfies ($\natural$), then so does $R_{(n)}$ for any $n > 0$;

(2) If R is standard graded and satisfies ($\natural$), then the following is true:

  (X) $R_{(n)}$ is nearly Gorenstein for any $n > s(R/\mathrm{tr}(\omega_R)_1 R)$;

  (Y) In particular, if R is nearly Gorenstein, then so is $R_{(n)}$ for any $n > 0$.

Thm B (M.) Assume that R is a semi-standard graded domain.

If R satisfies ($\natural$), then $\exists$ a integer $k_R \geq 0$ s.t. $R_{(n)}$ is nearly Gorenstein $\forall n > k_R$.

Rem 2

Thm A (2) (Y) fails if we consider semi-standard graded rings.

Indeed, there exists a nearly Gor semi-standard graded affine semigroup ring R whose 2-Veronese subring $R_{(2)}$ is not nearly Gorenstein.

Projective monomial curves satisfying (♮)

Let $\mathbf{a} = a_1, \cdots, a_n, a_{n+1} \in \mathbb{Z}_{>0}$ with $\gcd(a_1, \cdots, a_n, a_{n+1}) = 1$.

Notice that $\mathrm{pd}(\Bbbk[S_\mathbf{a}]) = n$ if $\Bbbk[S_\mathbf{a}]$ is CM.

Ex Let $n, k > 1$ and let $0 < j < (n+1)k$, $j \not\equiv 0 \pmod{k}$ be an integer.

- Put $\mathbf{a} = 1, 2, \cdots, n, n+1$.

  Then $R = \Bbbk[S_\mathbf{a}]$ is nearly Gorenstein with $h_{s(R)} = \mathrm{pd}(R)$.

- Put $\mathbf{a} = k, 2k, \cdots, nk, (n+1)k, j$.

  Then $R = \Bbbk[S_\mathbf{a}]$ is **not** nearly Gorenstein.

  However, R satisfies condition (♮) with $h_{s(R)} = \mathrm{pd}(R) - 1$.

Q. If R satisfies (♮) with $h_{s(R)} = \mathrm{pd}(R)$, then is it nearly Gorenstein?

R : CM projective monomial curve

Prop (M.)

- If R satisfies ($\natural$), then $h_{s(R)} \leq \mathrm{pd}(R)$.
- If R satisfies ($\natural$) with $h_{s(R)} = \mathrm{pd}(R)$, then R is nearly Gorenstein.

Cor (M.)

- If $\mathrm{pd}(R) = 2$, then R satisfies ($\natural$) $\Leftrightarrow$ R is nearly Gorenstein.
- If $\mathrm{pd}(R) = 3$, then TFAE:
  (1) R is not Gorenstein and nearly Gorenstein;
  (2) R satisfies ($\natural$) with $h_{s(R)} = \mathrm{pd}(R)$;
  (3) R satisfies ($\natural$) with $\mathrm{Hilb}(R, t) = \dfrac{1 + 3t + 3t^2 + \cdots + 3t^{s(R)}}{(1-t)^2}$.

Let R be a CM projective monomial curve.

- Let $r(R)$ denote the number of minimal generating system of ω_R.
- $h_{s(R)} \leq r(R)$ is always true.

Unsolved problem

If R satisfies ($\natural$), then $r(R) \leq \mathrm{pd}(R)$?

(1) Problem is true when R is nearly Gorenstein and $\mathrm{pd}(R) \leq 3$ or $h_{s(R)} = r(R)$.

(2) Even when $\mathrm{pd}(R) = 3$, the problem remains unsolved.

(3) If we change the assumption ($\natural$) into Gorenstein on the punctured spectrum, there exists a following example which satisfies $r(R) > \mathrm{pd}(R)$.

Ex

$R = \mathbb{Q}[S_{2,5,6,9}]$ is CM projective monomial curve with $\mathrm{tr}(\omega_R) = \mathfrak{m}^2$.

Thus R is Gorenstein on $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$, however, $r(R) = 5 > 3 = \mathrm{pd}(R)$.

Thank you!