Rook polynomials of almost symmetric Arf numerical semigroups

Meral Süer

Joint work with Mehmet Şirin Sezgin,
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Presentation Overview

1. Numerical Sets
2. Young diagram
3. Partitions
4. Rook polynomials
5. Rook polynomials of almost symmetric Arf numerical semigroups
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Numerical Sets I

- Let $\mathbb{Z}$ denote the set of integers, $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A subset S of $\mathbb{N}_0$ that contains zero and has finite complement in $\mathbb{N}_0$ is called a numerical set.

- A numerical set S is a numerical semigroup, if it satisfies that $x, y \in S \implies x + y \in S$.

- A numerical set S is considered proper if it is not equal to the set of nonnegative integers.

Numerical Sets II

Let us assume that S is indeed a proper numerical set.

- The complement of S within $\mathbb{N}_0$ is denoted by $\mathrm{G}(S)$.
- The elements of $\mathrm{G}(S)$ are called as the gaps of S.
- $|\mathrm{G}(S)| = \mathrm{g}(S)$ is called genus of S.
- $\max(\mathrm{G}(S)) = \mathrm{F}(S)$ is the Frobenius number of S.
- $\mathrm{F}(S) + 1 = \mathrm{C}(S)$ is the conductor of S.
- $S = \{0 = s_0, s_1, \ldots, s_{n-1}, s_n = \mathrm{C}(S), \rightarrow\}$, $s_{i-1} < s_i$ for $1 \leq i \leq n$; $0 = s_0 < s_1 < \cdots < s_{n-1}$ are called small elements of S. (Where "$\rightarrow$" means that all integers greater than $\mathrm{C}(S)$ belong to S .)

Numerical Sets III

Let S be a numerical semigroup.

- An integer x is a pseudo-Frobenius number of S if $x \notin S$ but $x + s \in S$ for all $s \in S \setminus \{0\}$. We will denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S,
- $|\mathrm{PF}(S)| = \mathrm{t}(S)$ is the type of S.

- Let S be a numerical semigroup. S is an almost symmetric semigroup if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{t}(S)}{2}$ [3].

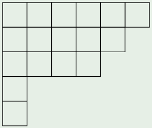
- A numerical semigoup is Arf if for all $x, y, z \in S$ with $x \geq y \geq z, \quad x + y - z \in S$[2].

Young diagram I

- A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.

Example 1

The picture depicted below is a Young diagram with 6 columns and 5 rows.



Young diagram II

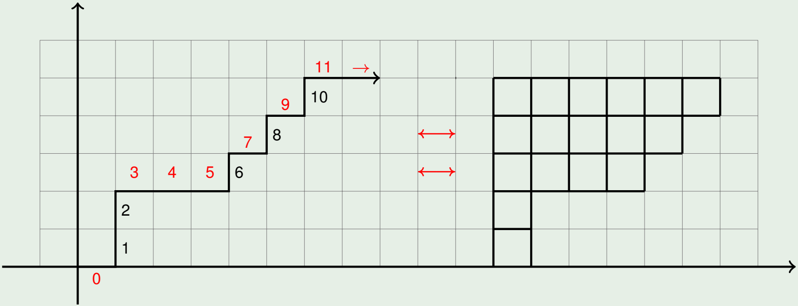
- Let S be a numerical set. We can construct a Young diagram Y_S corresponding to S by drawing a continuous polygonal path that starts from the origin in $\mathbb{Z}^2$. Starting with $s = 0$,
  1. if $s \in S$, draw a line of unit length to the right,
  2. if $s \notin S$, draw a line of unit length to up,

  and repeat it for $s + 1$. We continue this until $s = \mathrm{F}(S)$. The lattice restricted by this polygonal path, y axis and the horizontal line that is $\mathrm{g}(S)$ units above the origin defines the corresponding Young diagram Y_S.
  It is clear that every Young diagram corresponds to a unique proper numerical set. Thus the correspondence $\beta : \mathbb{S} \to \mathbb{Y}$, $\beta(S) = Y_S$ is a bijection between the collection $\mathbb{S}$ of proper numerical sets and the collection $\mathbb{Y}$ of Young diagrams.

Young diagram III

Example 2

Let $S = \{0, 3, 4, 5, 7, 9, 11 \rightarrow\}$ be a numerical set. The Young diagram corresponding to this numerical set is as follows.



Partitions I

- Given a positive integer N, a partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ of N is a non-increasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = N$.
- For each $i = 1, 2, \ldots, k$, the number λ_i is called a part of the partition.
- The number k of parts is called the length of the partition.
- If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ is a partition of N, then we write

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k] \vdash N.$$

Partitions II

- Given a partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k] \vdash N$, the Young diagram Y_λ corresponding to λ consists of k columns of boxes with lengths $\lambda_1, \lambda_2, \ldots, \lambda_k$.
- Clearly, every Young diagram represents a uniquely determined partition. Therefore, we get a bijection $\alpha : \mathbb{P} \to \mathbb{Y}$, $\alpha(\lambda) = Y_\lambda$, where $\mathbb{P}$ denotes the collection of all partitions and $\mathbb{Y}$ denotes the collection of all Young diagrams.

Example 3

The Young diagram in Example 1 corresponds to the partition $[5, 3, 3, 3, 2, 1] \vdash 17$.

Partitions III

- Let us note that the composition $\alpha^{-1}\beta$ is a bijection from the set $\mathbb{S}$ of proper numerical sets to the set $\mathbb{P}$ of partitions of positive integers: $\alpha^{-1}\beta : \mathbb{S} \to \mathbb{P}$, $\alpha^{-1}\beta(S) = \alpha^{-1}(Y_S)$.

Rook polynomials I

- Given a non-negative integer *m*, let [*m*] denote the set $\{1, 2, \ldots, m\}$.
- We define a board *B* with *m* rows and *n* columns to be a subset of [*m*]x[*n*].
- We call such a board an *mxn* board if *m* and *n* are the smallest such non-negative integer.
- Each of the elements in the board is referred to as a cell of the board.
- The set [*m*]x[*n*] is called the full *mxn* board [1].

Rook polynomials II

- The rook polynomial $R_B(x) = r_0(B) + r_1(B)x + \cdots + r_k(B)x^k + \ldots$ of a board B represents the number of ways that one can place various numbers of non-attacking rooks on B; i.e., no two rooks can lie in the same column or row.
- More specifically, $r_k(B)$ is equal to the number of ways of placing k non-attacking rooks on B.
- For any board, $r_0(B) = 1$ and $r_1(B)$ is equal to the number of cells in B.

Example 4

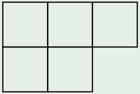


Figure: The Board B

For board B, $r_2(B) = 4$ as there are four different ways to place 2 non-attacking rooks on the board. It is not possible to place 3 or more rooks on this board. Hence the rook polynomial of this board is $R_B(x) = 1 + 5x + 4x^2$.

Rook polynomials of almost symmetric Arf numerical semigroups I

- If a numerical semigroup S is both almost symmetric semigroup and Arf, then S is called an almost symmetric Arf semigroup [5].

- Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ be a partition.
  1. If $\lambda = \alpha^{-1}\beta(S)$, for some almost symmetric semigroup S, then λ is called an almost symmetric partition.
  2. If $\lambda = \alpha^{-1}\beta(S)$, for some almost symmetric Arf semigroup S, then λ is called an almost symmetric Arf partition [5].

Rook polynomials of almost symmetric Arf numerical semigroups II

Theorem 5

Any almost symmetric Arf partition λ *is of the form either* $\lambda = [\alpha]$, *or* $\lambda = [\alpha + \beta, \alpha, \alpha - 1, \ldots, 2, 1]$, *where* $\beta \in \{1, 3, 5, \ldots, 2\alpha - 3, 2\alpha - 1, \rightarrow\}$, *for some* $\alpha \geq 1$ *[5].*

Remark 1

For $\alpha \geq 1$, *almost symmetric Arf semigroup belonging to the partition* $\lambda = [\alpha]$ *is of the form* $S = \{0, \alpha + 1, \rightarrow\}$.

Rook polynomials of almost symmetric Arf numerical semigroups III

Remark 2

For $\alpha \geq 1$ and $\beta \in \{1, 3, 5, \ldots, 2\alpha - 3, 2\alpha - 1, \rightarrow\}$, almost symmetric Arf semigroup belonging to the partition $\lambda = [\alpha + \beta, \alpha, \alpha - 1, ..., 2, 1]$ is of the form $S = \{0, \beta + 1, \beta + 3, \ldots, 2\alpha + \beta + 1, \rightarrow\}$.

Theorem 6

Let $S = \{0, \alpha + 1, \rightarrow\}$ be represented by an almost symmetric Arf semigroup as in Remark 1 and the Young diagram Y_S corresponding to S. Then the rook polynomial of Y_S is

$$R_{Y_S}(x) = 1 + \alpha x.$$

Rook polynomials of almost symmetric Arf numerical semigroups IV

Theorem 7

Let $C = \{0, \beta+1, \beta+3, \ldots, 2\alpha+\beta+1, \rightarrow\}$ *be represented by an almost symmetric Arf semigroup as in Remark 2 and the Young diagram* Y_C *corresponding to* C*. In this case, the rook polynomial of* Y_C *is*

$$R_{Y_C}(x) = c_0(Y_C) + c_1(Y_C)\,x + c_2(Y_C)\,x^2 + \cdots + c_{\alpha+1}(Y_C)\,x^{\alpha+1},$$

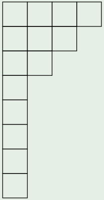
where $c_0(Y_C) = 1$, $c_1(Y_C) = \frac{\alpha(\alpha+1)}{2} + (\alpha+\beta)$, $c_{\alpha+1}(Y_C) = \beta$ *and for* $\mu = 2, \ldots, \alpha$

$$\begin{aligned} c_\mu(Y_C) &= \left[\sum_{\varepsilon_\mu=1}^{\alpha+1-\mu} \varepsilon_\mu \left(\sum_{\varepsilon_{\mu-1}=1}^{\varepsilon_\mu} \varepsilon_{\mu-1} \cdots \left(\sum_{\varepsilon_2=1}^{\varepsilon_3} \varepsilon_2 \left(\sum_{\varepsilon_1=1}^{\varepsilon_2} \varepsilon_1\right)\right) \cdots \right)\right] \\ &+ (\alpha+\beta-\mu+1)\left[\sum_{\varepsilon_{\mu-1}=1}^{\alpha+2-\mu} \varepsilon_{\mu-1} \left(\sum_{\varepsilon_{\mu-2}=1}^{\varepsilon_{\mu-1}} \varepsilon_{\mu-2} \cdots \left(\sum_{\varepsilon_2=1}^{\varepsilon_3} \varepsilon_2 \left(\sum_{\varepsilon_1=1}^{\varepsilon_2} \varepsilon_1\right)\right) \cdots \right)\right]. \end{aligned}$$

Rook polynomials of almost symmetric Arf numerical semigroups V

Example 8

For $\alpha = 3$ and $\beta = 5$, let's find the rook polynomial of the almost symmetric Arf semigroup $C = \{0, 6, 8, 10, 12, \rightarrow\}$. The Young diagram corresponding to C is shown below.



The rook polynomial of Y_C is $R_{Y_C}(x) = c_0(Y_C) + c_1(Y_C)x + c_2(Y_C)x^2 + c_3(Y_C)x^3 + c_4(Y_C)x^4$. Here $c_0(Y_C) = 1$, $c_1(Y_C) = \frac{3.4}{2} + (3+5) = 14$, and $c_4(Y_C) = 5$.

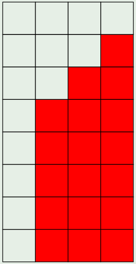
For $\mu = 1, 2, 3$ $\quad c_2(Y_C) = \left(\sum_{\varepsilon_2=1}^{2} \varepsilon_2 \left(\sum_{\varepsilon_1=1}^{\varepsilon_2} \varepsilon_1\right)\right) + 7\left(\sum_{\varepsilon_1=1}^{3} \varepsilon_1\right) = (1(1) + 2(1+2)) + 7(1+2+3) = 49$ and $c_3(Y_C) = \left(\sum_{\varepsilon_3=1}^{1} \varepsilon_3 \left(\sum_{\varepsilon_2=1}^{\varepsilon_3} \varepsilon_2 \left(\sum_{\varepsilon_1=1}^{\varepsilon_2} \varepsilon_1\right)\right)\right) + 6\left(\sum_{\varepsilon_2=1}^{2} \varepsilon_2 \left(\sum_{\varepsilon_1=1}^{\varepsilon_2} \varepsilon_1\right)\right) = (1(1(1))) + 6(1(1) + 2(1+2)) = 43$. Therefore, $R_{Y_C}(x) = 1 + 14x + 49x^2 + 43x^3 + 5x^4$.

Rook polynomials of almost symmetric Arf numerical semigroups VI

- The complement of a Young diagram is found by completing the rectangular grid with the length and width of the first row and first column respectively [4].

Example 9

The complement of Y_C given in Example 8 is shown below in red.



Rook polynomials of almost symmetric Arf numerical semigroups VII

Theorem 10

Let $C = \{0, \beta+1, \beta+3, \ldots, 2\alpha+\beta+1, \rightarrow\}$ *be an almost symmetric Arf semigroup as in Remark 2. The Young diagram corresponding to C is denoted by* Y_C *and the complement of* Y_C *is denoted by* $Y_{C'}$. *Then the rook polynomial of* $Y_{C'}$ *is* $R_{Y_{C'}}(x) = c_0(Y_{C'}) + c_1(Y_{C'})x + c_2(Y_{C'})x^2 + \cdots + c_\alpha(Y_{C'})x^\alpha$ *and the coefficients of* $R_{Y_{C'}}(x)$ *are as follows:*

$$c_j(Y_{C'}) = \begin{cases} 1 & \text{if } j = 0, \\ \sum_{\mu=0}^{j} h_\mu(H)\binom{\alpha-\mu}{j-\mu}\binom{\beta}{j-\mu}(j-\mu)! & \text{if } 1 \leq j \leq \alpha < \beta, \\ \sum_{\mu=j-\beta}^{j} h_\mu(H)\binom{\alpha-\mu}{j-\mu}\binom{\beta}{j-\mu}(j-\mu)! & \text{if } \beta \leq j \leq \alpha. \end{cases}$$

Where $h_0(H) = h_{\alpha-1}(H) = 1$ *and for* $\mu = 1, 2, \ldots, \alpha-2$

$$h_\mu(H) = \sum_{\varepsilon_\mu=1}^{\alpha-\mu} \varepsilon_\mu \left(\sum_{\varepsilon_{\mu-1}=1}^{\varepsilon_\mu} \varepsilon_{\mu-1} \cdots \left(\sum_{\varepsilon_2=1}^{\varepsilon_3} \varepsilon_2 \left(\sum_{\varepsilon_1=1}^{\varepsilon_2} \varepsilon_1 \right)\right) \cdots \right).$$

Rook polynomials of almost symmetric Arf numerical semigroups VIII

Example 11

Let's find the rook polynomial of the complement of Y_C given in Example 8. From Theorem 10, $h_0(H) = h_2(H) = 1$ and for $\mu = 1$, $h_1(H) = \sum_{\varepsilon_1=1}^{2} \varepsilon_1 = (1+2) = 3$. Then the rook polynomial of $Y_{C'}$ is
$R_{Y_{C'}}(x) = c_0(Y_{C'}) + c_1(Y_{C'})\,x + c_2(Y_{C'})\,x^2 + c_3(Y_{C'})\,x^3$ and the coefficients of $R_{Y_{C'}}(x)$ are as follows:

$c_0(Y_{C'}) = 1$ and for $1 \leq j \leq 3$

$c_1(Y_{C'}) = \sum_{\mu=0}^{1} h_\mu(H)\binom{3-\mu}{1-\mu}\binom{5}{1-\mu}(1-\mu)! = h_0(H)\binom{3}{1}\binom{5}{1}1! + h_1(H)\binom{2}{0}\binom{5}{0}0! = 1\binom{3}{1}\binom{5}{1}1! + 3\binom{2}{0}\binom{5}{0}0! = 18$

$c_2(Y_{C'}) = \sum_{\mu=0}^{2} h_\mu(H)\binom{3-\mu}{2-\mu}\binom{5}{2-\mu}(2-\mu)! = h_0(H)\binom{3}{2}\binom{5}{2}2! + h_1(H)\binom{2}{1}\binom{5}{1}1! + h_2(H)\binom{1}{0}\binom{5}{0}0! =$
$1\binom{3}{2}\binom{5}{2}2! + 3\binom{2}{1}\binom{5}{1}1! + 1\binom{1}{0}\binom{5}{0}0! = 91$

$c_3(Y_{C'}) = \sum_{\mu=0}^{3} h_\mu(H)\binom{3-\mu}{3-\mu}\binom{5}{3-\mu}(3-\mu)! = h_0(H)\binom{3}{3}\binom{5}{3}3! + h_1(H)\binom{2}{2}\binom{5}{2}2! + h_2(H)\binom{1}{1}\binom{5}{1}1! +$
$+h_3(H)\binom{0}{0}\binom{5}{0}0! = 1\binom{3}{3}\binom{5}{3}3! + 3\binom{2}{2}\binom{5}{2}2! + 1\binom{1}{1}\binom{5}{1}1! + 0\binom{0}{0}\binom{5}{0}0! = 125$

Therefore,

$$R_{Y_{C'}}(x) = 1 + 18x + 91x^2 + 125x^3.$$

References I

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Thanks

Questions? Comments?

meral.suer@batman.edu.tr