Orbit codes and lattices

Rameez Raja
rameeznaqash@nitsri.ac.in

Department of Mathematics
National Institute of Technology Srinagar, Srinagar-190006, Jammu and Kashmir, India

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JOINT WORK WITH

Sihem Mesnager
Department of Mathematics, University of Paris VIII, F-93526 Saint-Denis, Laboratory Analysis, Geometry and Applications, LAGA, University Sorbonne Paris Nord, CNRS, UMR 7539, F-93430, Villetaneuse, France and Telecom Paris, Polytechnic institute of Paris, 91120 Palaiseau, France

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Partitions, orbits and binary codewords

- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of of a positive integer $n \in \mathbb{Z}_{>0}$, denoted by, $\lambda \vdash n$, where $\lambda_1, \ldots, \lambda_r$ represents parts of the partition and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$

- To every partition of a positive integer, we can associate a finite abelian p-group of rank r, where r is the number of parts in the partition, that is, corresponding to a partition λ, a finite abelian p-group of rank r is given as,

$$A_{(p,\lambda)} = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\lambda_r}\mathbb{Z} \qquad (1)$$

- For distinct partitions and primes, a finite abelian group $\mathcal{G}$ is a direct sum of subgroups of type (1), that is,
$$\mathcal{G} = \bigoplus A_{(p,\lambda)}$$

- Consider a finite abelian p-group $\mathcal{G}_\eta = \mathbb{Z}/p^\eta\mathbb{Z}$ of rank one which corresponds to some part $\eta \in (\lambda_1, \ldots, \lambda_r)$ of the partition λ. Under the group action $Aut(\mathcal{G}_\eta) \times \mathcal{G}_\eta \longrightarrow \mathcal{G}_\eta$, orbits $\mathcal{O}_{1,\eta}, \mathcal{O}_{p,\eta}, \ldots, \mathcal{O}_{p^\eta,\eta}$ of elements of $\mathcal{G}_\eta$ are represented by $1, p, \ldots, p^\eta$, where $\mathcal{O}_{p^i,\eta} = \{p^i a : (a,p) = 1\}$, $0 \leq i \leq \eta$, (a,p) denotes the gcd of positive integers a and p.

- We follow the next procedure to construct a binary codeword of the type $0^{t_1}1^{r_1}0^{t_2}1^{r_2}\ldots 0^{t_h}1^{r_h}$ or $1^{t_1}0^{r_1}1^{t_2}0^{r_2}\ldots 0^{t_h}1^{r_h}$ from $\mathcal{G}_\eta$ called as *automorphism orbit codeword* . The powers t_i and r_i, $1 \leq i \leq h$, of 0 and 1 bit strings are determined by the structure of $\mathcal{G}_\eta$ and the action $Aut(\mathcal{G}_\eta) \times \mathcal{G}_\eta \longrightarrow \mathcal{G}_\eta$.

1. If $xy \not\equiv 0(mod\ p^\eta)$, then assign a bit 0 to all elements x and y of some orbit in the collection $\{\mathcal{O}_{1,\eta}, \mathcal{O}_{p,\eta}, \ldots, \mathcal{O}_{p^\eta,\eta}\}$ of orbits of elements of $\mathcal{G}_\eta$.
2. If $xy \equiv 0(mod\ p^\eta)$, then assign a bit 1 to all elements x and y of some orbit in the collection $\{\mathcal{O}_{1,\eta}, \mathcal{O}_{p,\eta}, \ldots, \mathcal{O}_{p^\eta,\eta}\}$ of orbits of elements of $\mathcal{G}_\eta$.

- Let ϕ denotes the Euler's totient function, and let $a_1, \ldots, a_{\phi(p^\eta)}$ be positive integers which are relatively prime with p^η. The description of two cases we consider for the part η of λ is as follows.

- **Case-I:** If $\eta = 2k, k \in \mathbb{Z}_{>0}$, then orbits of the group action $Aut(\mathcal{G}_\eta) \times \mathcal{G}_\eta \longrightarrow \mathcal{G}_\eta$ are listed as follows,

$$\begin{aligned}
\mathcal{O}_{1,\eta} &= \{a_1, \ldots, a_{\phi(p^\eta)}\},\\
\mathcal{O}_{p,\eta} &= \{pa_1, \ldots, pa_{\phi(p^{\eta-1})}\},\\
&\vdots\\
\mathcal{O}_{p^k,\eta} &= \{p^k a_1, \ldots, p^k a_{\phi(p^{\eta-k})}\},\\
&\vdots\\
\mathcal{O}_{p^{\eta-1},\eta} &= \{p^{\eta-1} a_1, \ldots, p^{\eta-1} a_{\phi(p)}\},\\
\mathcal{O}_{p^\eta,\eta} &= \{p^\eta\}.
\end{aligned}$$

- A positive integer k is the minimum power of p such that $xy \equiv 0(mod\ p^{\eta})$ for all $x, y \in \mathcal{O}_{p^k,\eta}$. So we associate a bit string of $1's$ with the orbit $\mathcal{O}_{p^k,\eta}$. The power of the bit string is equal to the cardinality of $\mathcal{O}_{p^k,\eta}$, and this bit string represents an initial bit string of the intended binary codeword c_η which we construct from $\mathcal{G}_\eta$. Furthermore, $k-1$ is the maximum power of p such that for all $x, y \in \mathcal{O}_{p^{k-1},\eta}$, $xy \not\equiv 0(mod\ p^{\eta})$. Consequently, we associate a string of $0's$ with the orbit $\mathcal{O}_{p^{k-1},\eta}$. Note that the power of this bit string is the cardinality of $\mathcal{O}_{p^{k-1},\eta}$, and it represents another part of c_η.

- Next, we attach a bit string of $1's$. The $1's$ in a bit string correspond to elements of the orbit $\mathcal{O}_{p^{k+1},\eta}$, and $k+1$ is the minimum power of p such that for all $x \in \mathcal{O}_{p^{k-1},\eta}$ and $y \in \mathcal{O}_{p^{k+1},\eta}$, $xy \equiv 0(mod\ \ p^{\eta})$.

- We alternate attaching bit strings of $1's$ and $0's$ to get the desired binary codeword c_{η} from $\mathcal{G}_{\eta}$. This process is exhausted when the sum of powers of bit strings is the order of the group $\mathcal{G}_{\eta}$. Thus for a group $\mathcal{G}_{\eta}$, the *automorphism orbit codeword* is given as,

$$1^{\phi(p^k)}0^{\phi(p^{k-1})}1^{\phi(p^{k+1})}0^{\phi(p^{k-2})}\dots 0^{\phi(1)}1. \qquad (2)$$

- The sum of powers of bit strings is,

$$p^{\eta} - p^{\eta-1} + p^{\eta-1} - p^{\eta-2} + \cdots + 2 - 1 + 1 = p^{\eta} = |\mathcal{G}_{\eta}|.$$

- **Case-II.** $\eta = 2k-1$, $k \in \mathbb{Z}_{>0}$. We can list orbits of the group action $Aut(\mathcal{G}_{\eta}) \times \mathcal{G}_{\eta} \longrightarrow \mathcal{G}_{\eta}$ in the same manner as we did in case-I.

- However, in this case, we cannot begin the construction of c_{η} from a bit string of $1's$, since from the structure of $\mathcal{G}_{\eta}$, k is the least integral power of p such that for all $x \in \mathcal{O}_{p^{k-1},\eta}$ and $y \in \mathcal{O}_{p^{k},\eta}$ the relation $xy \equiv 0 (mod\ \ p^{\eta})$ holds.

- Again as above, $k-1$ is the maximum power of p such that for all $x, y \in \mathcal{O}_{p^{k-1},\eta}$, $xy \not\equiv 0(mod\ p^{\eta})$. So the initial bit string of c_η consists of $0's$, which correspond to elements of the orbit $\mathcal{O}_{p^{k-1},\eta}$.

- The next bit string of $1's$ in c_η correspond to elements of the orbit $\mathcal{O}_{p^k,\eta}$. Continue the same process of adding alternate bit strings of $0's$ and $1's$ we obtain the automorphism orbit codeword c_η of $\mathcal{G}_\eta$ given by,

$$0^{\phi(p^{k-1})}1^{\phi(p^k)}0^{\phi(p^{k-2})}\dots 0^{\phi(1)}1. \tag{3}$$

- As in **Case-I**, the sum of powers of bit strings is the order of $\mathcal{G}_\eta$. Observe that there is one to one correspondence between orbits $\mathcal{O}_{1,\eta}$, $\mathcal{O}_{p,\eta}$, $\ldots$, $\mathcal{O}_{p^\eta,\eta}$ and bit strings $(b_{1,\eta})$, $(b_{p,\eta})$, $\ldots$, $(b_{p^\eta,\eta})$ of 0s and 1s. Therefore, the cardinality of any orbit equals the number of bits in the corresponding bit string of 0s or 1s.

- Fix some partition λ. Let c_η be an automorphism orbit codeword of some constituent of $A_{p,\lambda}$. Corresponding to some orbit $\mathcal{O}_{p^t,\eta}$, there is bit string $b_{p^t,\eta}$, where $0 \leq t \leq \eta$. Furthermore, let $\mu \neq \eta$ be another part of λ. By $b_{p^t,\eta} \hookrightarrow b_{p^l,\mu}$, we mean $b_{p^t,\eta}$ is a sub bit string of $b_{p^l,\mu}$, $0 \leq l \leq \mu$. Equivalently, $\mathcal{O}_{p^t,\eta} \hookrightarrow \mathcal{O}_{p^l,\mu}$ indicates that the correspondence between orbits $\mathcal{O}_{p^t,\eta}$ and $\mathcal{O}_{p^l,\mu}$ is one to one and $\mathcal{O}_{p^t,\eta} \subseteq \mathcal{O}_{p^l,\mu}$. If for each t and l, $b_{p^t,\eta} \hookrightarrow b_{p^l,\mu}$, then we write $c_\eta \hookrightarrow c_\mu$.

- Let $\mathcal{H}$ be some group. Then a homomorphism $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$, defines a codeword c_φ as a vector $c_\varphi = (\varphi(s_1), \varphi(s_2) \ldots \varphi(s_k))$, where $\varphi(s_i)$ is the image of $s_i \in \mathcal{S}$, $1 \leq i \leq k$, $\mathcal{S}$ is a fixed set of generators of $\mathcal{G}$.

- The set $Hom(\mathcal{G}, \mathcal{H})$ of all homomorphisms between groups $\mathcal{G}$ and $\mathcal{H}$ can be viewed as error-correcting codes. More specifically, a *homomorphism code* is defined as the set of all homomorphisms from $\mathcal{G}$ to $\mathcal{H}$, denoted by, $\mathcal{C} = Hom(\mathcal{G}, \mathcal{H})$.

- Note that the codeword c_φ of a *homomorphism code* $\mathcal{C} = Hom(\mathcal{G}, \mathcal{H})$ is specified by the image of generators of a group $\mathcal{G}$. In contrast, automorphism orbit codewords are based on elements of $Hom(\mathcal{G}, \mathcal{G})$, partitions and $Aut(\mathcal{G})$-orbits of the group action $Aut(\mathcal{G}) \times \mathcal{G} \longrightarrow \mathcal{G}$.

- So, automorphism orbit codewords are generalized homomorphism codewords which provides an interesting interplay of partitions, orbits of group action and binary codewords.
- In [1, 2, 5], the authors have discussed interesting generation of some graphs by binary generating codes of the type $0^{s_1}1^{r_1}0^{s_2}1^{r_2}\dots0^{s_k}1^{r_k}$, where s_i, r_i, $1 \leq i \leq k$, are some positive integers. They have determined some fascinating algebraic and combinatorial invariants from powers s_i and r_i of bits 0 and 1 involved in $0^{s_1}1^{r_1}0^{s_2}1^{r_2}\dots0^{s_k}1^{r_k}$.

Hasse diagram with points as binary bit strings

- Now, we begin to establish a poset structure of automorphism orbit codewords.

- Let $|\mathcal{G}| = n$ and $|\mathcal{H}| = m$. Consider the group action $Aut(\mathcal{G}) \times \mathcal{G} \longrightarrow \mathcal{G}$. Let $\mathcal{O}_{g_1,n}, \mathcal{O}_{g_2,n}, \ldots, \mathcal{O}_{g_k,n}$ denotes the $Aut(\mathcal{G})$-orbits, where $k \leq n$ and $g_1, g_2, \ldots, g_k$ are representatives of these orbits.

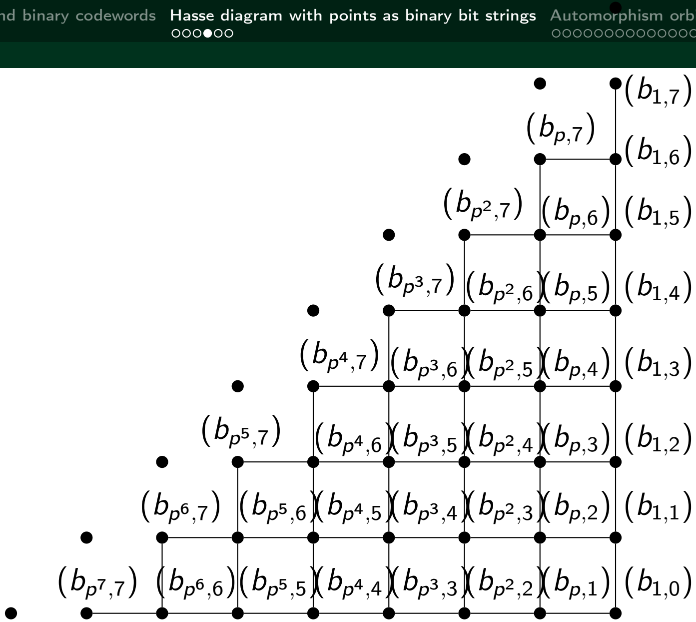
- A homomorphism $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ is said to be an *orbit cover* of $\mathcal{G}$ if for each i, $1 \leq i \leq k$, $\varphi(\mathcal{O}_{g_i,n}) \subseteq \mathcal{O}_{\varphi(g_i),m}$, $\varphi(g_1), \varphi(g_2), \ldots, \varphi(g_k)$ are representatives of orbits of the action $Aut(\mathcal{H}) \times \mathcal{H} \longrightarrow \mathcal{H}$. Note that for some subset $\mathcal{U} \subseteq \mathcal{G}$, $\varphi(\mathcal{U}) = \{\varphi(u) : u \in \mathcal{U}\}$.

Here is our first result.

Proposition 1: Let $\mathcal{G}$ and $\mathcal{H}$ be two finite abelian p-groups of rank one such that $|\mathcal{G}| = p^{\eta}$ and $|\mathcal{H}| = p^{\mu}$. If $b_{p^t,\eta}$ and $b_{p^l,\mu}$ represents bit strings of codewords c_{η} and c_{μ}, then $\mathcal{G}$ admits an orbit cover ($\hookrightarrow$) if and only if $t \leq l$ and $\eta - t \geq \mu - l$.

- Denote by $\mathcal{S}_{\mu} = \{(b_{p^t,\mu}) : 0 \leq t \leq \mu\}$, a set of bit strings of a codeword associated with $\mathbb{Z}/p^{\mu}\mathbb{Z}$ and let $\mathcal{S} = \bigsqcup_{\mu \in \mathbb{Z}_{>0}} \mathcal{S}_{\mu}$ be the disjoint union over all $\mu \in \mathbb{Z}_{>0}$.

- By **Proposition 1**, one can verify that the relation of "orbit cover" between bit strings of automorphism orbit codewords is reflexive and transitive in fact it is a partial order on set $\mathcal{S}$. Moreover, the relation "orbit cover" is independent of the parity of the power (part of a partition) of a prime discussed in equations (2) and (3). Below, we present a pictorial representation (poset realization) of $\mathcal{S}$ with respect to the partial order "orbit cover". Nodes in some part of a poset is labelled by bit strings as shown in Figure 1.



$(b_{1,7})$
$(b_{p,7})$
$(b_{1,6})$
$(b_{p^2,7})$ $(b_{p,6})$ $(b_{1,5})$
$(b_{p^3,7})$ $(b_{p^2,6})$ $(b_{p,5})$ $(b_{1,4})$
$(b_{p^4,7})$ $(b_{p^3,6})$ $(b_{p^2,5})$ $(b_{p,4})$ $(b_{1,3})$
$(b_{p^5,7})$ $(b_{p^4,6})$ $(b_{p^3,5})$ $(b_{p^2,4})$ $(b_{p,3})$ $(b_{1,2})$
$(b_{p^6,7})$ $(b_{p^5,6})$ $(b_{p^4,5})$ $(b_{p^3,4})$ $(b_{p^2,3})$ $(b_{p,2})$ $(b_{1,1})$
$(b_{p^7,7})$ $(b_{p^6,6})$ $(b_{p^5,5})$ $(b_{p^4,4})$ $(b_{p^3,3})$ $(b_{p^2,2})$ $(b_{p,1})$ $(b_{1,0})$

Figure 1

- A bit string $(b_{1,0})$ corresponds to the orbit $\mathcal{O}_{1,0}$ which consists of zero element only, that is, $\mathcal{O}_{1,0}$ is an orbit of the group action $Aut(G) \times G \to G$, where $G = \{0\}$.

- Thus given a partition λ, a binary code generated by automorphism orbit codewords (which we discuss in a subsequent section) can be derived through a particular construction process involving the poset $\mathcal{S}$ and orbits of the group action $Aut(A_{(p,\lambda)}) \times A_{(p,\lambda)} \to A_{(p,\lambda)}$.

- The automorphism groups of r constituents $\mathbb{Z}/p^{\lambda_1}\mathbb{Z}, \mathbb{Z}/p^{\lambda_2}\mathbb{Z}, \ldots, \mathbb{Z}/p^{\lambda_r}\mathbb{Z}$ of a finite abelian p-group $A_{p,\lambda}$ contributes to group actions,

$$Aut(\mathbb{Z}/p^{\lambda_1}\mathbb{Z}) \times \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^{\lambda_1}\mathbb{Z}, \ldots, Aut(\mathbb{Z}/p^{\lambda_r}\mathbb{Z}) \times \mathbb{Z}/p^{\lambda_r}\mathbb{Z} \longrightarrow \mathbb{Z}/p^{\lambda_r}\mathbb{Z}.$$

- Consequently, there are orbits, and we consider the following product of the product of orbits,

$$\prod \left(\prod_{i=1}^{\lambda_1} \mathcal{O}_{p^i,\lambda_1} \prod_{i=1}^{\lambda_2} \mathcal{O}_{p^i,\lambda_2} \cdots \prod_{i=1}^{\lambda_r} \mathcal{O}_{p^i,\lambda_r} \right). \qquad (4)$$

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Automorphism orbit codes and lattice structure

- Given a partition, we define $\mathcal{C}_\lambda = (c_{\lambda_1}, c_{\lambda_2}, \ldots, c_{\lambda_r})$, a code which we refer as *an automorphism orbit code* associated with $A_{(p,\lambda)}$ (viewed as an induced subposet of $\mathcal{S}$). $\mathcal{C}_\lambda$ is generated by automorphism orbit codewords c_{λ_i}, $1 \leq i \leq r$, which in turn are generated by bit strings described in the preceding section.

- Variable length code (VLC) is called entropy coding (data compression), a technique where each event is assigned a codeword with a different number of bit strings. Observe that $\mathcal{C}_\lambda$ is a variable length code since a fixed number of source symbols (orbits) are encoded into a variable number of out symbols (bit strings).

- Notice that the codeword length in $\mathcal{C}_\lambda$ depends on the source symbol's property. A significant advantage of VLC is that it does not degrade the signal quality. Much literature is available where VLC has been studied for data compression and signal processing.

- From the structure of $\mathcal{S}$, we immediately view an *ideal* I_λ of $\mathcal{C}_\lambda$ generated by some bit strings of codewords of $\mathcal{C}_\lambda$, that is, I_λ as a code is generated by bit strings which correspond to elements of the product (4).

- In the following result, we make use of the partial order "$\hookrightarrow$" to establish a relation between order ideals I_λ and $I_{\lambda'}$ of $\mathcal{S}$.

Theorem 1: For any two partitions λ and λ', $A_{(p,\lambda)}$ admits an orbit cover if and only if $I_{\lambda'} \cup \mathcal{I} \subseteq I_\lambda$, where $\mathcal{I} \subset Hom(A_{(p,\lambda)}, A_{(p,\lambda')})$.

- It is known that there is a one-to-one correspondence between antichains and ideals, namely, the maximal elements of an ideal of a poset form an antichain and generate the ideal (see Section 3.1, [4]). Furthermore, we have the following observation.

Remark: For any partition λ, $\mathcal{C}_\lambda$ is an induced subset of $\mathcal{S}$. If $\mathfrak{I}$ denotes a code generated by bit strings of some automorphism orbit codes in $\mathcal{S}$ such that $\mathfrak{I}$ is an ideal of $\mathcal{S}$, then $\mathfrak{I} \cap \mathcal{C}_\lambda$ is always an ideal of $\mathcal{C}_\lambda$. Note that $\mathfrak{I}$ is generated by bit strings corresponding to distinct automorphism orbit codewords of $\mathcal{C}_\lambda$. Suppose $\mathfrak{I}$ is generated by maximal bit strings of the code $\mathcal{C}_\lambda$. Then by [4] (see Section 3.1), there is one-to-one correspondence between a code $\mathfrak{I}$ and a code represented by $\mathfrak{I} \cap \mathcal{C}_\lambda$.

- Now, it is natural to observe orbits of the group action $\mathcal{G}_{p,\lambda} \times A_{p,\lambda} \longrightarrow A_{p,\lambda}$, where $\mathcal{G}_{p,\lambda}$ is an automorphism group of $A_{p,\lambda}$. Note that $\mathcal{G}_{p,\lambda}$ acts on each of the r constituents of $A_{p,\lambda}$. So similar to (4) the orbits of the action are given as,

$$\prod \left(\prod_{i=1}^{\lambda_1} \tilde{\mathcal{O}}_{p^i,\lambda_1} \prod_{i=1}^{\lambda_2} \tilde{\mathcal{O}}_{p^i,\lambda_2} \cdots \prod_{i=1}^{\lambda_r} \tilde{\mathcal{O}}_{p^i,\lambda_r} \right). \qquad (5)$$

- $\tilde{\mathcal{O}}_{p^z,\lambda_i}$, $1 \leq z \leq \lambda_i$, $1 \leq i \leq r$, are $\mathcal{G}_{p,\lambda}$-orbits of the group action $\mathcal{G}_{p,\lambda} \times A_{p,\lambda} \longrightarrow A_{p,\lambda}$. Define a set $\tilde{\mathcal{S}}_\lambda = \{\tilde{b}_{p^z,\lambda_i} : 0 \leq z \leq \lambda_i, 1 \leq i \leq r\}$ to be the set of bit strings of automorphism orbit codewords $\tilde{c}_{\lambda_1}, \tilde{c}_{\lambda_2}, \ldots \tilde{c}_{\lambda_r}$

- Note that bit strings $\tilde{b}_{p^z,\lambda_i}, 0 \leq z \leq \lambda_i, 1 \leq i \leq r$, correspond to $\mathcal{G}_{p,\lambda}$-orbits $\tilde{\mathcal{O}}_{p^z,\lambda_i}$ of r constituents of $A_{p,\lambda}$.

- For $\tilde{b}_{p^z,\lambda_i}, \tilde{b}_{p^z,\lambda_j} \in \tilde{\mathcal{S}}_\lambda$ say $\tilde{b}_{p^z,\lambda_i} \hookrightarrow\!\!\!\!\twoheadrightarrow \tilde{b}_{p^z,\lambda_j}$ if $\tilde{b}_{p^z,\lambda_i}$ is strict substring of $\tilde{b}_{p^z,\lambda_j}$, that is, if $\tilde{b}_{p^z,\lambda_i}$ is a substring of $\tilde{b}_{p^z,\lambda_j}$ but $\tilde{b}_{p^z,\lambda_j}$ is not a substring of $\tilde{b}_{p^z,\lambda_i}$.

- Equivalently, $\tilde{\mathcal{O}}_{p^z,\lambda_i} \hookrightarrow\!\!\!\!\twoheadrightarrow \tilde{\mathcal{O}}_{p^z,\lambda_j}$ if $\tilde{\mathcal{O}}_{p^z,\lambda_i}$ is a subset of $\tilde{\mathcal{O}}_{p^z,\lambda_j}$ but $\tilde{\mathcal{O}}_{p^z,\lambda_j}$ is not a subset of $\tilde{\mathcal{O}}_{p^z,\lambda_i}$. If the relation "$\hookrightarrow\!\!\!\!\twoheadrightarrow$" holds for all $\mathcal{G}_{p,\lambda}$-orbits of $A_{p,\lambda}$, then we say that "$\hookrightarrow\!\!\!\!\twoheadrightarrow$" is a *strict orbit cover* of $A_{p,\lambda}$.

- It is easy to very that the relation "$\hookrightarrow\!\!\!\!\twoheadrightarrow$" on the set $\tilde{\mathcal{S}}_\lambda$ is a partial order. This implies that $\tilde{\mathcal{S}}_\lambda$ is a partially ordered set with respect to "$\hookrightarrow\!\!\!\!\twoheadrightarrow$".

- Let $\mathcal{L}_\lambda = \{I_\lambda : \lambda \text{ is a partition}\}$ be a collection of order ideals associated with automorphism orbit codes $\mathcal{C}_\lambda$ of $\mathcal{S}$. Clearly, $\mathcal{L}_\lambda$ is a lattice. The interesting relation between binary codes of $\tilde{\mathcal{S}}$ and order ideals in $\mathcal{L}_\lambda$ is that $\tilde{\mathcal{S}}$ is isomorphic as a poset to $\mathcal{L}_\lambda$. The map $\Phi : \tilde{\mathcal{S}} \longrightarrow \mathcal{L}_\lambda$ given by $\Phi(\tilde{C}_\lambda) = I_\lambda$ exhibits an isomorphism between these posets.

- For some $n \in \mathbb{Z}_{>0}$, let $\mathrm{Y}(n) = \{\lambda : \lambda \vdash n\}$ be the set of all partitions of n. We relate two partitions $\mu = (\mu_1, \ldots, \mu_s), \lambda = (\lambda_1, \ldots \lambda_r) \in \mathrm{Y}(n)$ as, $\mu \leqslant \lambda$ if $\mu \subset \lambda$, that is, if μ is contained in λ. One can easily verify that the set $\mathrm{Y}(n)$ for the relation "$\leqslant$" is a locally finite distributive lattice with the smallest unique element as $\bar{0}$, the empty set.

- Y(n) is called as *Young's lattice* or the lattice of *Young diagrams*, since to every partition of n there is a Young diagram associated to it. Denote the young diagram of λ by Y_λ.

- Notice that if $\mu \leqslant \lambda$, then for automorphism orbit codes $\tilde{\mathcal{C}}_\mu, \tilde{\mathcal{C}}_\lambda \in \tilde{\mathcal{S}}$, $\tilde{c_{\mu_i}} \hookrightarrow\!\!\!\!\rightarrow \tilde{c_{\lambda_j}}$ holds for each i and j, $1 \leq i \leq s$ and $1 \leq j \leq r$.

- The length of a codeword $\tilde{c_{\lambda_j}}$ is the number of bit strings in $\tilde{c_{\lambda_j}}$. Note that in Y_λ there are λ_1 boxes in the top row of Y_λ, λ_2 boxes in the second last row from the top of Y_λ and so on.

- It follows that the length of $\tilde{c_{\lambda_j}}$ equals the number of boxes in the j-th row of Y_λ. Thus, there is a correspondence between partitions of n and codes of $\tilde{\mathcal{S}}$. If $\tilde{\mathcal{S}}_n$ denotes the set of all automorphism orbit codes corresponding to all partitions of n, then $\tilde{\mathcal{S}}_n$ is an induced poset of $\tilde{\mathcal{S}}$. In particular, $\tilde{\mathcal{S}}_n$ is a locally finite distributive lattice, and each code in $\tilde{\mathcal{S}}_n$ can be identified as a Young diagram associated with a partition. The following statement holds.

Theorem 2: For some $\lambda \vdash n$, the bijection $\tilde{\mathcal{C}}_\lambda \longrightarrow \lambda$ is a poset isomorphism from $\tilde{\mathcal{S}}_n$ to $\mathrm{Y}(n)$.

Preparing a binary coding set up for future work on Numerical Semigroups, potentially in collaboration with Maria!!

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