The multiples of a numerical semigroup

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Introduction

A **numerical semigroup** S is a subset of $\mathbb{N}$ closed under addition of natural numbers, containing $\{0\}$ and such that $\mathbb{N} \setminus S$ has finite cardinality.

Let $S \neq \mathbb{N}$ be a numerical semigroup.

The **Frobenius number** of S, denoted by $\mathrm{F}(S)$, is the maximum of $\mathbb{N} \setminus S$. The **multiplicity** of S, $\mathrm{m}(S)$, is the minimum of $S \setminus \{0\}$.

The numerical semigroup S has a unique minimal system of generators, denoted $\mathrm{msg}(S)$; its cardinality is the so-called **embedding dimension** of S, denoted by $\mathrm{e}(S)$.

Roughly speaking, the *Frobenius problem* is to find formulas of $\mathrm{F}(S)$ in terms of $\mathrm{msg}(S)$.

Introduction

Let T be a numerical semigroup. Given $d \in \mathbb{N} \setminus \{0\}$, we write

$$\frac{T}{d} = \{x \in \mathbb{N} \mid d\, x \in T\}.$$

which is also a numerical semigroup called the **quotient of T by d**.

Definition

Let S and T be numerical semigroups and $d \in \mathbb{N} \setminus \{0\}$. We say that T is a $d-$**multiple of** S if $\frac{T}{d} = S$.

Correspondingly, given two numerical semigroups S and T, we say that T is a multiple of S if there exists $d \in \mathbb{N} \setminus \{0\}$ such that $\frac{T}{d} = S$.

Introduction

Proposition

Let S and T be two numerical semigroups and $d \in \mathbb{N} \setminus \{0\}$. Then T is $d-$multiple of S, that is,

$$\frac{T}{d} := \{x \in \mathbb{N} \mid d\,x \in T\} = S,$$

if and only if

$$d\,(\mathbb{N} \setminus S) \subseteq \mathbb{N} \setminus T \subseteq \mathbb{N} \setminus d\,S.$$

Specifically, in this case, $\mathrm{F}(T) \geq d\,\mathrm{F}(S)$.

(Mostly open) Problems

Let $S \neq \mathbb{N}$ be a numerical semigroup and $d \in \mathbb{N} \setminus \{0\}$.

- Compute (if possible) all $d-$multiples of S.
- Solve the Frobenius problem for $d-$multiples of S in terms of $\mathrm{F}(S)$.
- Determine if a given numerical semigroup has a multiple with embedding dimension greater than or equal to three.
- Determine

$$\min\{\mathrm{e}(T) \mid T \text{ is a multiple of } S\}$$

  for a given numerical semigroup S.

This number is called **quotient rank of** S.

The quotient rank of S is bounded above by $\mathrm{e}(S)$, since $S/1 = S$. Then, S is said to have **full quotient rank** when the quotient rank of S is equal to $\mathrm{e}(S)$.

Computation of all $d-$multiples of S

We write $\mathrm{M}_d(S)$ for the set of all numerical semigroups $d-$multiples of S, that is,

$$\mathrm{M}_d(S) = \{ T \text{ is a numerical semigroup } \mid d\,(\mathbb{N} \setminus S) \subseteq \mathbb{N} \setminus T \subseteq \mathbb{N} \setminus d\,S \}.$$

It is known [Rosales+García-Sánchez (2008), Swanson (2009)] that there are infinitely many elements in the set $\mathrm{M}_d(S)$.

We write $\max \mathrm{M}_d(S)$ for the set of maximal elements of $\mathrm{M}_d(S)$ with respect to inclusion.

Proposition

- *The set* $\max \mathrm{M}_d(S)$ *is never empty.*
- *If* $T \in \max \mathrm{M}_d(S)$ *then* $\mathrm{F}(T) = d\,\mathrm{F}(S)$.

In particular, $\max \mathrm{M}_d(S)$ *has finite cardinality.*

Example: $\mathrm{F}(T) = d\,\mathrm{F}(S) \not\Rightarrow T \in \max \mathrm{M}_d(S)$

Let $S = \langle 3, 4, 5\rangle$ and $d = 3$. If $T = \langle 4, 7, 9, 10\rangle$ and $T' = \langle 4, 5, 7\rangle$, then one can see that $T \subsetneq T'$, that

$$\frac{T}{d} = \frac{T'}{d} = S$$

and that

$$\mathrm{F}(T) = \mathrm{F}(T') = 3\,\mathrm{F}(S) = 6.$$

Therefore, $T \notin \max \mathrm{M}_d(S)$ although has the minimum possible Frobenius number among the elements of $\mathrm{M}_d(S)$.

Moreover, one can easily check[1] that

$$\max \mathrm{M}_d(S) = \{\langle 4, 5, 7\rangle\}$$

because $\langle 4, 5, 7\rangle$ is the only irreducible numerical semigroup with Frobenius number equal to 6.

[1] GAP Package NumericalSgps: `https://gap-packages.github.io/numericalsgps/`

Recall that a numerical semigroup with Frobenius number F is **irreducible** if and only if it is maximal in the set of all numerical semigroups with Frobenius number F.

Proposition

S is irreducible if and only if every $T \in \max \mathrm{M}_d(S)$ is irreducible.

Since $\max \mathrm{M}_d(S)$ is a subset of the set of (irreducible, if S is) numerical semigroups with Frobenius number $d\,\mathrm{F}(S)$, we can naively compute $\max \mathrm{M}_d(S)$.

Computation of all $d-$multiples of S (cont.)

Theorem

There exists an explicit surjective map

$$\Theta_S^d : \mathrm{M}_d(S) \longrightarrow \max \mathrm{M}_d(S);\ \ T \mapsto \Theta_S^d(T)$$

Therefore, to compute $\mathrm{M}_d(S)$, it is "enough" to know what the fibers of Θ_S^d are like.

Proposition

Given $R \in \max \mathrm{M}_d(S)$, *the set* $(\Theta_S^d)^{-1}(R)$ *can be arranged (by inclusion) as a rooted three* $\mathcal{G}(R)$ *with root* R.

The set of children of $T \in \mathcal{G}(R)$ - Case 1.

Proposition

Let $T \in \mathrm{M}_d(S)$ and $R = \Theta_S^d(T)$. If $\mathrm{F}(T) = d\,\mathrm{F}(S)$, then the set of children of T in $\mathcal{G}(R)$ is equal to the union of

$$\{T \setminus \{x\} \subset \mathbb{N} \mid x \in \mathrm{msg}(T), x \notin d\,S \text{ and } x > \mathrm{F}(T)\}$$

and

$$\{T \setminus \{x\} \subset \mathbb{N} \mid x \in \mathrm{msg}(T), x \notin d\,S,\ x < \mathrm{F}(T) \text{ and } x = (*)\},$$

where

$$(*) = \max\Big\{z \in \{x\} \cup (\mathbb{N} \setminus T) \mid 2z \in T \setminus \{x\}, z \notin d\,(\mathbb{N} \setminus S) \text{ and } z - y \notin T \setminus \{x\},\ \text{for every } y \in T \setminus \{x\}\Big\}.$$

The set of children of $T \in \mathcal{G}(R)$ - Case 2.

Proposition

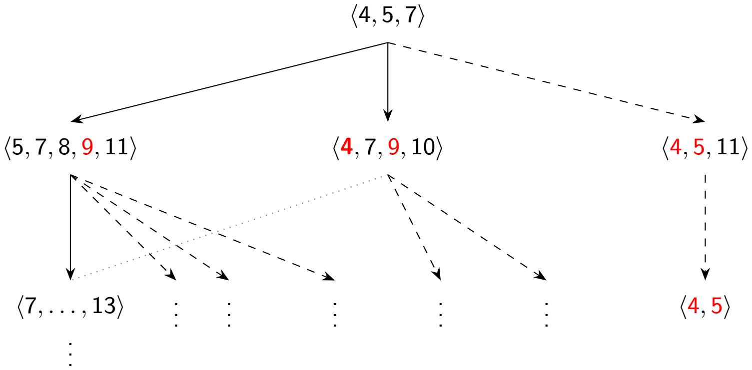
Let $T \in \mathsf{M}_d(S)$ and $R = \Theta_S^d(T)$. If $\mathsf{F}(T) > d\,\mathsf{F}(S)$, then the set of children of T in $\mathcal{G}(R)$ is equal to

$$\{T \setminus \{x\} \subset \mathbb{N} \mid x \in \mathsf{msg}(T), x \notin d\,S \text{ and } x > \mathsf{F}(T)\}.$$

In particular, if $\mathsf{F}(T) \neq d\,\mathsf{F}(S)$ and $x < \mathsf{F}(T)$ for every $x \in \mathsf{msg}(T)$, then T is a leaf of $\mathcal{G}(R)$.

Example

Let $S = \langle 3, 4, 5 \rangle$ and $d = 3$. In this case, $\max \mathrm{M}_d(S) = \{\langle 4, 5, 7 \rangle\}$ and the rooted tree grows as depicted below:



(*) Continuous arrows connect those numerical semigroups in $\mathrm{M}_d(S)$ with minimum possible Frobenius number.

$\mathrm{M}_d(S)-$system of generators

Proposition

$T \in \mathrm{M}_d(S)$ if and only if there exists a finite subset X of S such that

- $\langle X \rangle \cap d(\mathbb{N} \setminus S) = \varnothing$,
- $\gcd(X \cup \{d\}) = 1$,
- $T = \langle X \rangle + d\,S$.

In this case, if there is not proper subset of X with that property, we say that X a is **minimal $\mathrm{M}_d(S)-$system of generators of T.**

Theorem

If $T \in \mathrm{M}_d(S)$, then $\mathrm{msg}(T) \cap (\mathbb{N} \setminus d\,\mathrm{msg}(S))$ is the (unique) minimal $\mathrm{M}_d(S)-$system of generators of T.

The $\mathrm{M}_d(S)-$**embedding dimension** of $T \in \mathrm{M}_d(S)$ is the cardinality of its minimal $\mathrm{M}_d(S)-$system of generators of T.

$\mathrm{M}_d(S)$−embedding dimension one

Proposition

A subset T of $\mathbb{N}$ is a d−multiple of S with $\mathrm{M}_d(S)$−embedding dimension one if and only if there exists $x \in S$ with $\gcd(x, d) = 1$ such that

$$T = \langle x \rangle + d\,S.$$

In this case, $x = \min(T \setminus dS)$.

Recall that a numerical semigroup T is a **gluing of T_1 and T_2** if $T = \lambda T_1 + \mu T_2$ for some $\lambda \in T_1 \setminus \mathrm{msg}(T_1)$ and $\mu \in T_2 \setminus \mathrm{msg}(T_2)$ with $\gcd(\lambda, \mu) = 1$.

Corollary

A subset T of $\mathbb{N}$ is a d−multiple of S with $\mathrm{M}_d(S)$−embedding dimension one with $\min(T \setminus dS) \notin \mathrm{msg}(S)$ if and only if T is a gluing of $\mathbb{N}$ and S.

$\mathrm{M}_d(S)-$embedding dimension one (cont.)

Proposition (Frobenius problem)

If T is a $d-$multiple of S with $\mathrm{M}_d(S)-$embedding dimension one, then

$$\mathrm{F}(T) = (d-1)\min(T \setminus dS) + d\,\mathrm{F}(S).$$

Formulas for the genus, pseudo-Frobenius numbers are also obtained.

Full quotient rank

Recall that S has full quotient rank if

$$\min\{\mathrm{e}(T) \mid T \text{ is a multiple of } S\} = \mathrm{e}(S).$$

Proposition

If $\mathrm{msg}(S) = \{a_1, \ldots, a_e\}$ *and*

$$\sum_{\substack{j=1\\ j\neq i}}^{e} a_j - a_i \notin S \text{ for every } i \in \{1, \ldots, e\},$$

then S has full quotient rank.

Corollary (Numerical semigroups having unique Betti element)

If $c_1, \ldots, c_e$ *are relatively prime integers greater than one and*

$$\mathrm{msg}(S) = \left\{ \prod_{j=1, j\neq i}^{e} c_j \mid i \in \{1, \ldots, e\} \right\},$$

then S has full quotient rank.

Full quotient rank

Recall that S has full quotient rank if

$$\min\{\mathrm{e}(T) \mid T \text{ is a multiple of } S\} = \mathrm{e}(S).$$

Proposition

If $\mathrm{msg}(S) = \{a_1, \ldots, a_e\}$ *and*

$$\sum_{\substack{j=1 \\ j \neq i}}^{e} a_j - a_i \notin S \text{ for every } i \in \{1, \ldots, e\},$$

then S has full quotient rank.

Open question. Is the above condition necessary?

Proposition

If $\mathrm{msg}(S) = \{a_1 < \ldots < a_e\}$ *and* $\sum_{\substack{j=1 \\ j \neq i}}^{e} a_j - a_i \notin S$ *for every* $i \in \{1, \ldots, e\}$*, then*

$$a_1 \geq 2^{e-1}.$$

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Thanks for your attention!