Special Subdiagrams of Young Diagrams

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Outline

Numerical Sets

Notations 1

- $\mathbb{N}$ *is the set of non-negative integers.*
- S *is a* **numerical set**, *i.e.* $S \subseteq \mathbb{N}$, $0 \in S$ *and* $|\mathbb{N} \setminus S| < \infty$.
- $\mathrm{G}(S) = \mathbb{N} \setminus S$ *is the set of* **gaps** *of* S.
- $\mathrm{g}(S) = |\,\mathrm{G}(S)|$ *is the* **genus** *of* S.
- $\mathrm{F}(S)$ *is the* **Frobenius number** *of* S.
- $\mathrm{C}(S)$ *is the* **conductor** *of* S.

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Definitions 2

- A numerical set S is called proper if $S \neq \mathbb{N}$.
- The elements in a proper numerical set S that are less than $\mathrm{C}(S)$ are called the **small elements** of S.
- For an element $s \in S$, the set difference $S - s$ is defined as follows:

$$S - s = \{x - s \mid x \in S \text{ and } x \geq s\}.$$

Basic Properties 3

- *If S contains n small elements* $0 = s_0 < s_1 < \cdots < s_{n-1}$ *in ascending order, we write* $S = \{0, s_1, \ldots, s_{n-1}, s_n = C(S), \rightarrow\}$.

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- *If* $s \geq \mathrm{C}(S)$, *then* $S - s = \mathbb{N}$.
- *However, if* s *is equal to a small element* $s_i \in S$, *then* $S - s$ *can be expressed as:*

$$S - s = \{0 = s_i - s, s_{i+1} - s, \ldots, s_n - s, \rightarrow\}.$$

In this case, $S - s$ *is a numerical set with* $\mathrm{F}(S - s) = \mathrm{F}(S) - s$, $\mathrm{C}(S - s) = \mathrm{C}(S) - s$ *and* $\mathrm{G}(S - s) = \{g - s \mid g \in \mathrm{G}(S)$ *and* $g > s\}$*.*

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Example 4

$S = \{0, 4, 5, 7, 8, 10, 12, \rightarrow\}$ is a numerical set whose $\mathrm{F}(S) = 11$, $\mathrm{C}(S) = 12$, $\mathrm{G}(S) = \{1, 2, 3, 6, 9, 11\}$ and $\mathrm{g}(S) = 6$.

Young Diagrams

Definitions 5

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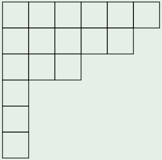
Example 6

The picture depicted below is a Young diagram with 6 columns and 6 rows.



Young diagrams and numerical sets

The correspondence between Young diagrams and numerical sets

- For a numerical set S, we can create a Young diagram Y_S that corresponds to S by sketching a continuous polygonal path starting from the origin in $\mathbb{Z}^2$. We begin with $s = 0$, and for each subsequent s we do the following:
  1. if $s \in S$, then we draw a unit-length line to the right.
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- This procedure establishes a bijection between the set of proper numerical sets and the set of Young diagrams.

Young diagrams and partitions

Partitions

For a positive integer N, a **partition** λ is a non-increasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = N$, denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

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The correspondence between Young diagrams and partitions

- For a Young diagram, listing all the lengths of each column gives a partition. Conversely, every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ corresponds to a Young diagram with λ_1 rows and n columns where the lengths of columns are $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively.

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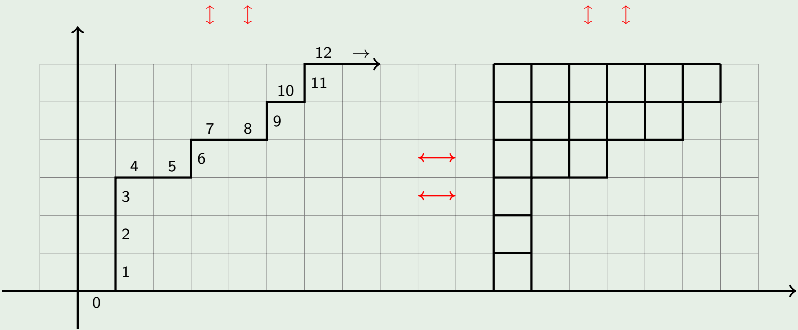
- This correspondence is also a bijection between the set of partitions and the set of Young diagrams.

Example 7

$S = \{0, 4, 5, 7, 8, 10, 12, \rightarrow\} \longleftrightarrow \lambda = (6, 3, 3, 2, 2, 1)$



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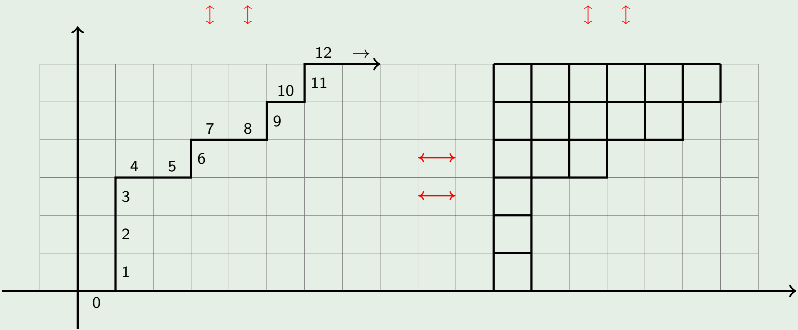

Subdiagrams

Let Young diagrams Y_λ and Y_μ correspond to the partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, respectively. We say that Y_μ is a **subdiagram** of Y_λ and we write $Y_\mu \subseteq Y_\lambda$ if $\mu_i \leq \lambda_i$ for each $i = 1, 2, \ldots, n$.

Special Subdiagrams

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Let Y be a Young diagram. The following process gives a new Young diagram which has less rows and columns than Y.

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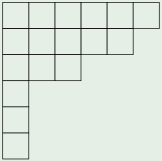
2. Consider each column of Y, excluding the first one, and add one unit-box just beneath each column that did not receive a box under in the previous step.
3. Delete the columns on the left of the box added to the far left and most bottom, and delete the rows above the box added to the rightmost and very top.

This Young diagram is called the **first special diagram** of Y.

Example 9

Let Y be the Young diagram as in Example 6 having 6 rows and 6 columns. We get its first special diagram as follows;



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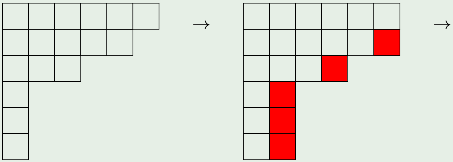

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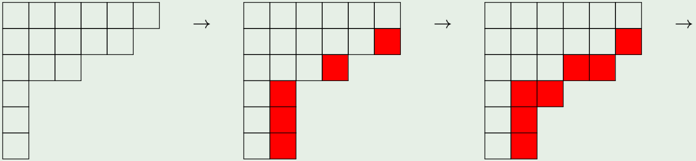

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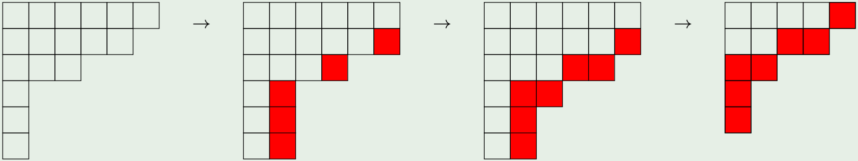

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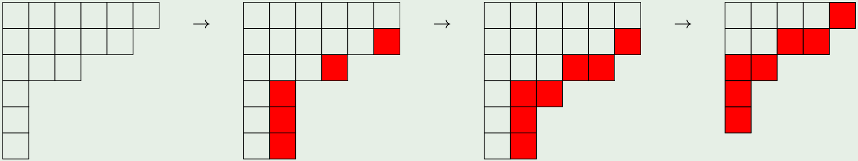
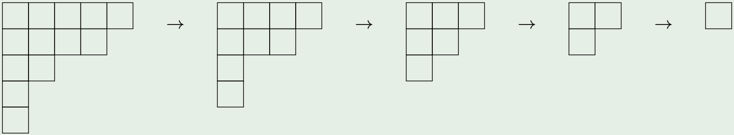


Then we get consecutive special diagrams of Y as follows;



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A Young diagram whose bottom row has length one is called a **reduced** Young diagram.

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Let Y be a Young diagram whose bottom row has length a, and let $Y^\star$ denote the Young diagram obtained by deleting the first $a-1$ columns of Y. The way we define $Y^\star$ shows that Y is reduced if and only if $Y^\star = Y$. It is easy to see that special diagrams of Y and $Y^\star$ are completely the same. Therefore, to find the special diagrams of Y we can consider $Y^\star$.

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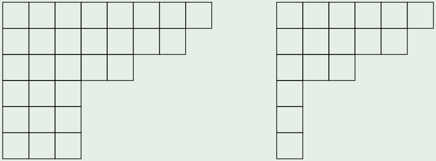
Example 11

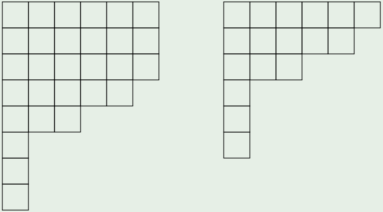
Let us consider the following two Young diagrams. Their special diagrams are all the same.



Likewise, if the rightmost column of a Young diagram Y has length b, to find its special diagrams we can consider the Young diagram obtained by deleting the top $b-1$ rows of Y. Special diagrams in this scenario remain identical as well.

Example 12

Let us consider the following Young diagrams. We can easily find out their special diagrams are all the same.



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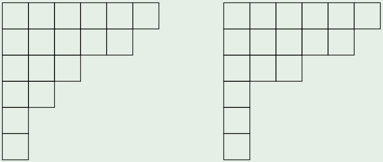
Example 13

Let us consider the following Young diagrams. If we construct their first special diagrams we can see that they are identical, and so are the others.



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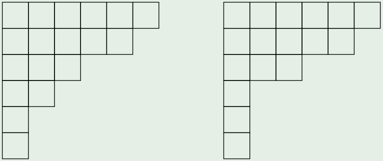

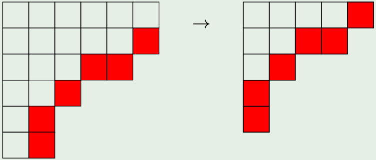

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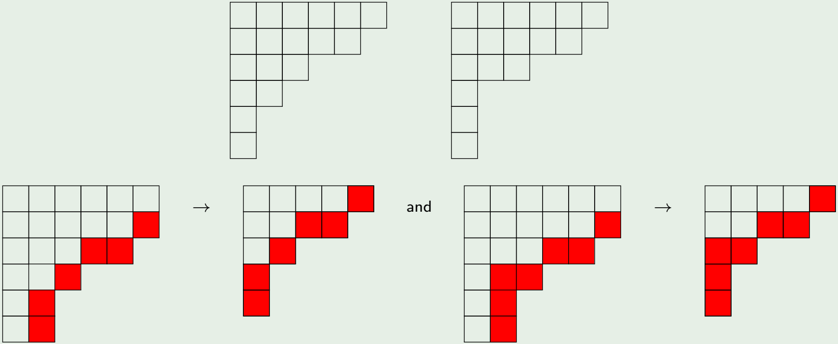

Proposition 14

The first special diagram of a Young diagram Y with n columns is a subdiagram of Y, and all special diagrams gives a sequence of subdiagrams of Y with length smaller than n.

Therefore, from now on, we will refer to special diagrams of a Young diagram as special subdiagrams.

Special subdiagrams and numerical sets

Definition 15

A numerical set S is called reduced if its corresponding Young diagram Y_S is reduced.

Proposition 16

A numerical set S is reduced if and only if $1 \notin S$.

Definition 17

A numerical set S closed under addition is called a numerical semigroup.

Corollary 18

All numerical semigroups except $\mathbb{N}$ are reduced.

Proposition 19

Let $S = \{0, s_1, \ldots, s_n, \rightarrow\}$ *be a numerical set and* s_r *be the smallest element of* S *where* $s_{r+1} - s_r \neq 1$*. Then the numerical set* $S^\star$ *corresponding to* $(Y_S)^\star$ *is* $S - r$*.*

Proposition 19

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Proposition 20

Let $S = \{0 = s_0, s_1, \ldots, s_n, \rightarrow\}$ be a reduced numerical set and Y_S be its Young diagram. If Y_T is the first special subdiagram of Y_S, then the corresponding numerical set to Y_T is $T = \{0 = t_0, t_1, \ldots, t_{n-1}, \rightarrow\}$ where each t_i is characterized as follows.

1. $t_0 = s_0 = 0.$
2. For $i = 1, 2, \ldots, n-2$, $t_i = \begin{cases} s_i & \text{if } s_{i+1} - s_i \neq 1, \\ s_i - 1 & \text{if } s_{i+1} - s_i = 1. \end{cases}$
3. $t_{n-1} = s_{n-1}.$

Proposition 21

Let $S = \{0 = s_0, s_1, \dots, s_n, \rightarrow\}$ *be a reduced numerical set and* Y_S *be its Young diagram where the gap set of* S *is* $\mathrm{G}(S) = \{g_1, \dots, g_t\}$*. If* T *is the numerical set corresponding to the first special subdiagram of* Y_S*, then for some* $k \in \{1, 2, \dots, t-1\}$*,*

1. $g_{k+1} - g_k \leq 2 \implies g_k \in \mathrm{G}(T)$,
2. $g_{k+1} - g_k > 2 \implies g_k \notin \mathrm{G}(T)$ *and* $g_{k+1} - 2 \in \mathrm{G}(T)$,
3. $g_{k+1} > s_{n-1} \implies g_r \notin \mathrm{G}(T)$ *for all* $r > k$,

where $\mathrm{G}(T)$ *is the gap set of* T*.*

Theorem 22

Let $S = \{0 = s_0, s_1, \ldots, s_n, \rightarrow\}$ *be a numerical semigroup and* Y_S *be its Young diagram. Let* Y_T *be the first special subdiagram of* Y_S*, and* T *be the corresponding numerical set to* Y_T*. Then* T *is a numerical semigroup if and only if for any non-zero elements* $x, y \in T$ *we have* $x + y \geq t_{n-1}$*, or* $x + y < t_{n-1}$ *and there exist an element* $s_k \in S$ *for some* $k \in \{1, 2, \ldots, n-2\}$ *where one of the following conditions hold*

1. $x + y = s_k$ *and* $s_k + 1 \notin S$,
2. $x + y = s_k$ *and* $s_k + 1, s_k + 2 \in S$,
3. $x + y = s_k - 1$ *and* $s_k + 1 \in S$,
4. $x + y = s_k - 2$ *and* $s_k - 1 \in S$,
5. $x + y = s_k - 2$ *and* $s_k - 2 \in S$.

Special subdiagrams of some numerical semigroups

Definition 23

A numerical set S is **symmetric** if and only if for each $k \in \{0, 1, 2, \ldots, \mathrm{F}(S)\}$ exactly one of k and $\mathrm{F}(S) - k$ is an element of S, and that a Young diagram is called **symmetric** if its corresponding numerical set is symmetric.

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Notation 24

$\mathrm{PF}(S) = \{f \in \mathrm{G}(S) | f + s \in S$ *for all* $s \in S\}$ *is the set of* **pseudo-Frobenius** *numbers, and* $\mathrm{t}(S) := |\,\mathrm{PF}(S)|$ *is the* **type** *of* S.

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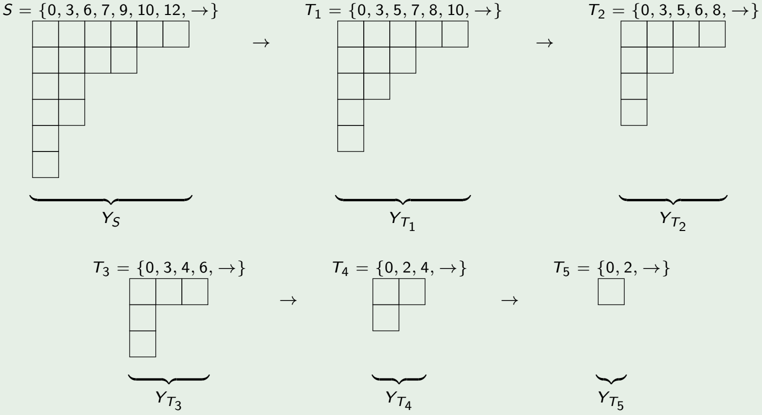
Definition 25

If a numerical semigroup S satisfies the property that $2\,\mathrm{g}(S) = \mathrm{F}(S) + \mathrm{t}(S)$, then it is called an **almost symmetric** numerical semigroup.

Example 26

Consider now the numerical semigroup $S = \{0, 3, 6, 7, 9, 10, 12, \rightarrow\}$ which is symmetric, and its Young diagram Y_S. We find the special subdiagrams of Y_S and their corresponding numerical sets as follows:

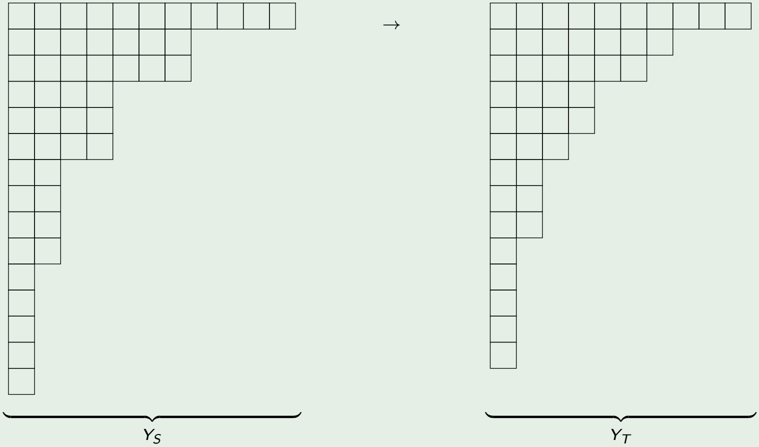


Notice that each T_i is symmetric, but T_1 is not a numerical semigroup.

Example 27

$S = \{0, 6, 11, 12, 16, 17, 18, 21, 22, 23, 24, 26, \rightarrow\}$
$\mathrm{G}(S) = \{1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 17, 19, 23\}$
$\mathrm{PF}(S) = \{5, 10, 15, 20, 25\}$ and $2\,\mathrm{g}(S) = \mathrm{F}(S) + \mathrm{t}(S)$

$T = \{0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, \rightarrow\}$
$\mathrm{G}(T) = \{1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 17, 19, 23\}$
$\mathrm{PF}(T) = \{14, 19, 23\}$ and $2\,\mathrm{g}(T) \neq \mathrm{F}(T) + \mathrm{t}(T)$

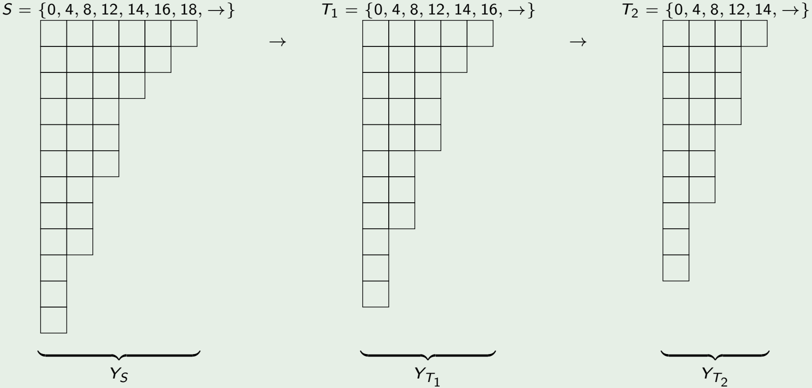


Here S is an almost symmetric numerical semigroup. On the other hand, T is not almost symmetric although it is a numerical semigroup.

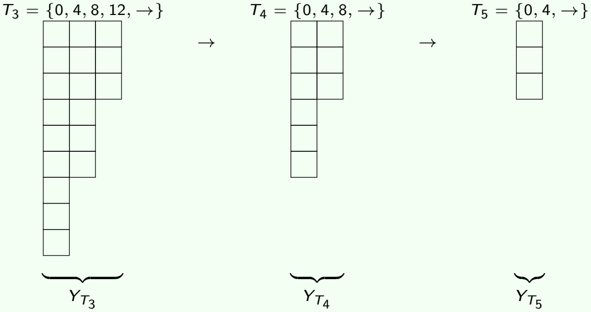
A numerical semigroup S is **Arf** if for every $s_1, s_2, s_3 \in S$ with $s_1 \leq s_2 \leq s_3$ we have the property that $s_2 + s_3 - s_1 \in S$.

Example 28

Consider the Arf numerical semigroup $S = \{0, 4, 8, 12, 14, 16, 18, \rightarrow\}$ and its Young diagram Y_S. We list all the special subdiagrams of Y_S and their corresponding numerical sets as follows:



Example 28



Notice that each T_i is Arf.

Proposition 29

Let S be a numerical set and Y_S be its Young diagram. Let Y_T be the first special subdiagram of Y_S and T be its corresponding numerical set.

1. *If S is an Arf numerical semigroup, then T is an Arf numerical semigroup.*
2. *If S is symmetric, then T is symmetric.*

Proposition 29

Let S be a numerical set and Y_S be its Young diagram. Let Y_T be the first special subdiagram of Y_S and T be its corresponding numerical set.

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Remark 30

Example 27 shows that given an almost symmetric numerical semigroup the corresponding numerical set to the first special subdiagram of its Young diagram does not have to be almost symmetric even if it is a numerical semigroup.

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Thank you for your attention!