Studying the Buchweitz Set of Numerical Semigroups

S. Eliahou
J.I. García-García
Daniel Marín-Aragón
A. Vigneron-Tenorio

Jerez, July 2024

Introduction

García-García, J.I., Marín-Aragón, D., Torres, F., Vigneron-Tenorio, A. (2021). On reducible non-Weierstrass semigroups. Open Mathematics. DOI: https://doi.org/10.1515/math-2021-0111

Introduction

Notation

$nG = \{g_1 + \ldots + g_n \mid g_1, \ldots, g_n \in G\}$.

Example

- $G = \{2, 3, 5, 6, 7\} = [2, 3] \sqcup [5, 7]$
- $2G = \{4, 5, 7, 8, 9, 6, 8, 9, 10, 10, 11, 12, 12, 13\}$.
- $2G = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$.
- $2G = [4, 13]$

Introduction

Definition

S is an n-Buchweitz semigroup for some $n \geq 2$ if

$$|nG| > (2n-1)(g-1)$$

.

Theorem (Buchweitz)

All n-Buchweitz semigroups are non-Weierstrass.

Introduction

Definition

Let $G \subset \mathbb{Z}$ be a finite subset. We define the funcion $\beta_G : \mathbb{N}_+ \to \mathbb{Z}$ as

$$\beta_G(n) = |nG| - (2n-1)(|G|-1).$$

Notation

$$\mathcal{B}(G) = \{n \geq 2 \mid \beta_G(n) \geq 1\}.$$

Introduction

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Example

- $G = \emptyset$.
- $|G| = |nG| = 0$.
- $\beta_G(n) = 2n - 1, \ n \geq 1$.
- $\mathcal{B}(G) = 2 + \mathbb{N}$

Introduction

$$\beta_G(n) = |nG| - (2n-1)(|G|-1).$$

$$\mathcal{B}(G) = \{n \geq 2 \mid \beta_G(n) \geq 1\}.$$

Example

- $|G| = 1.$
- $|nG| = 1.$
- $\beta_G(n) = 1,\ n \geq 1.$
- $\mathcal{B}(G) = 2 + \mathbb{N}$

Introduction

$$\beta_G(n) = |nG| - (2n-1)(|G|-1).$$

$$\mathcal{B}(G) = \{n \geq 2 \mid \beta_G(n) \geq 1\}.$$

Example

- $S = \langle 3, 7 \rangle$
- $G = \{1, 2, 4, 5, 8, 11\}$.
- $|G| = 6$.
- $2G = [2, 10] \cup [12, 13] \cup [15, 16] \cup \{19, 22\}$.
- $|2G| = 15$.
- $\beta_G(n) = 0,\ n \geq 1$.
- $\mathcal{B}(G) = \emptyset$

Introduction

Proposition (Buchweitz)

S Weierstrass $\Rightarrow \mathcal{B}(\mathbb{N} \setminus S) = \emptyset$.

$$\beta_G(n) = |nG| - (2n-1)(|G|-1).$$

Example (Buchweitz)

- $S = \langle 13, 14, 15, 16, 17, 18, 20, 22, 23 \rangle$
- $G = [1, 12] \sqcup \{19, 21, 24, 25\}$.
- $|G| = 16$.
- $2G = [2, 50] \setminus \{39, 41, 47\}$.
- $|2G| = 46$.
- $\beta_G(2) = 46 - 3 \cdot 15 = 1$.

Technical results

Theorem (Nathason)

Let $G \subset \mathbb{N}$ be a finite subset of cardinality $k \geq 2$ containing 0 and such that $\gcd(G) = 1$. Let $f = max(G)$. Then there exist integers c, d and subsets $C \subset [0, c-2]$, $D \subset [0, d-2]$ such that

$$nG = C \sqcup [c, fn-d] \cup (fn - D)$$

for all $n \geq \max\{(|G|-2)(f-1)f, 1\}$.

Corollary

Let $G \subset \mathbb{N}_+$ be a finite subset containing $\{1, 2\}$. Let $f = \max(G)$. Then there is an integer $b < 1$ such that

$$|nG| = (f-1)n + b$$

for all $n \geq (|G|-2)(f-2)(f-1)$.

Asymptotic Behaviour of β

Theorem

Let $G \subset \mathbb{N}_+$ be a finite set containing $\{1, 2\}$. Let $f = \max(G)$ and $g = |G|$. Then

$$\lim_{n \to \infty} \beta_G(n) = \begin{cases} -\infty & \text{if } f \leq 2g - 2, \\ +\infty & \text{if } g \geq 2g. \end{cases}$$

Finally, if $f = 2g - 1$, then $\beta_G(n)$ is constant and nonpositive for n large enough.

Sketch of the proof

- $\beta_G(n) = |nG| - (2n - 1)(|G| - 1)$.
- $\beta_G(n) = (f - 2g + 1)n + b + 1 - g$.

Application to Numerical Semigroups

Definition

Let S be a numerical semigroup. We define the Buchweitz set of S as $\mathrm{Buch}(S) = \mathcal{B}(\mathbb{N} \setminus S)$.

Proposition

Let S be a numerical semigroup with Frobenius number f and genus $g \geq 1$. Then $f \leq 2g - 1$.

Theorem

Let S be a numerical semigroup of genus $g \geq 2$. Then $\mathrm{Buch}(S)$ is finite.

Application to Numerical Semigroups

Sketch of the proof

- $|G| = g \geq 2$; $f = \max(G)$.
- $m \geq 2$; $f \leq 2g - 1$.
- If $m = 3$ then $\{1, 2\} \in G$.
- If $m = 2$ then $S = \langle 2, b \rangle$ for $b \geq 5$.
  - Komeda (1998), Olivera (1991).
  - $G = \{1, 3, \ldots, b - 2\}$.
  - $G - 1 = \{0, 2, \ldots, b - 3\}$.
  - $(G - 1)/2 = [0, (b - 3)/2]$
  - $|nG| = |n((G - 1)/2)| = n(|G| - 1) + 1$.
  - $\beta_G(n) = -(n - 1)|G| + n \leq 0$.
  - $\mathrm{Buch}(S) = \mathcal{B}(G) = \emptyset$.

Unboundedness of $|\mathrm{Buch}(S)|$

Proposition

For any integer $b \geq 3$, there exists a numerical semigroup such that $\mathrm{Buch}(S) = [2, b]$.

Sketch of the proof

- $k = b - 2$.
- $m = 6k + 15$.
- $q = 2$.
- $G = [1, m-1] \sqcup \{2m-7, 2m-5, 2m-2, 2m-1\}$.

More intervals

Proposition

Let $k \geq 1$. Let S be the numerical semigroup, of multiplicity $m = 6k + 19$ and depth $q = 2$ with gapset

$$G = [1, m-1] \sqcup \{2m-7, 2m-6, 2m-2, 2m-1\}.$$

Then $\mathrm{Buch}(S) = [3, k+3]$.

More intervals

Proposition

Let $k \geq 1$ and $i \in \{1, 2, 3\}$, let S_i be the numerical semigroup with gapset:

$$G_1 = [1, m_1 - 1] \sqcup \{2m_1 - 6, 2m_1 - 2, 2m_1 - 1\},$$

$$G_2 = [1, m_2 - 1] \sqcup \{2m_2 - 10, 2m_2 - 4, 2m_2 - 3, 2m_2 - 1\},$$

$$G_3 = [1, m_3 - 1] \sqcup \{2m_3 - 10, 2m_3 - 9, 2m_3 - 2\},$$

where $m_1 = 4k + 22$, $m_2 = 7k + 44$ and $m3 = 5k + 55$, respectively.
Then $\text{Buch}(S_1) = [4, k + 4]$, $\text{Buch}(S_2) = [5, k + 5]$ and $\text{Buch}(S_3) = [6, k + 6]$.

More intervals

Proposition

For any integer $k \geq 1$, there is a numerical semigroup S such that $\text{Buch}(S) = [7 + 2k, 7 + 4k]$.

Sketch of the proof

The corresponding gapset is:
$G = [1, 43 + 27k + 4k^2] \cup \{80 + 51k + 8k^2, 85 + 53k + 8k^2, 86 + 53k + 8k^2\}$.

Homework

Open questions

- Let $G \subset \mathbb{N}$ a gapset. Is $\mathcal{B}(G)$ an interval of integers?
- Let $T \subset 2 + \mathbb{N}$ be any finite subset. Does there exist a gapset $G \subset \mathbb{N}$, such that $\mathcal{B}(G) = T$?

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Thank you very much!