The ideals of a numerical semigroup with embedding dimension two

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$I(\langle a, b \rangle)$-semigroups

Outline

Some algorithms on $I(\langle a, b\rangle)$-semigroups

Principal $\mathrm{I}(\langle a, b\rangle)$-semigroups

$\mathrm{I}(\langle a, b\rangle)$-semigroups in general

$\mathrm{I}(\langle a, b\rangle)$-semigroups with ideal dimension two

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Some algorithms on $I(\langle a, b\rangle)$-semigroups

1. S numerical semigroup, algorithm to decide if S is an $I(\langle a, b\rangle)$-semigroup.
2. If S is a numerical semigroup, an algorithmic procedure to compute the set $\{\{a, b\} \subseteq \mathbb{N} \mid \gcd\{a, b\} = 1 \text{ and } S \text{ is an } I(\langle a, b\rangle) \text{ semigroup}\}$.

Algorithm to determine if S is an $I(\langle a, b \rangle)$-semigroup

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Proposition: Let S and T be numerical semigroups. Then T is an $\mathrm{I}(S)$-semigroup if and only if $\mathrm{msg}(T) + \mathrm{msg}(S) \subseteq T$ and $\mathrm{msg}(T) \subseteq S$.

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Proposition: Let S and T be numerical semigroups. Then T is an $\mathrm{I}(S)$-semigroup if and only if $\mathrm{msg}(T) + \mathrm{msg}(S) \subseteq T$ and $\mathrm{msg}(T) \subseteq S$.

Algorithm:
INPUT: A numerical semigroup S.
OUTPUT: TRUE, if S is an $\mathrm{I}(\langle a, b\rangle)$-semigroup, and FALSE otherwise.

1) If $\mathrm{msg}(S) \not\subseteq \langle a, b\rangle$, then return FALSE.
2) If $\mathrm{msg}(S) + \{a, b\} \not\subseteq S$, then return FALSE.
3) Return TRUE.

Example:

Is $S = \langle 4, 9, 14, 19 \rangle$ an $\mathrm{I}(\langle 4, 5 \rangle)$-semigroup?.

1) $\mathrm{msg}(S) = \{4, 9, 14, 19\} \subseteq \langle 4, 5 \rangle$.
2) $\{4, 9, 14, 19\} + \{4, 5\} = \{8, 9, 13, 14, 18, 19, 23, 24\} \subseteq S$.
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3) FALSE.

Algorithm: S n. s., $\{\{a,b\} \subseteq \mathbb{N} \mid S$ is an $\mathrm{I}(\langle a,b\rangle)$-semigroup$\}$.

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Proposition: If S is a numerical semigroup and $S \neq \mathbb{N}$, then $S \cup \mathrm{PF}(S)$ is also a numerical semigroup.

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Theorem: A numerical semigroup S is an $\mathrm{I}(\langle a,b\rangle)$-semigroup if and only if one of the following conditions is verified:

1) $S = \langle a,b\rangle$.
2) $\mathrm{msg}(S) \subseteq \langle a,b\rangle$ and $\{a,b\} \subseteq \mathrm{PF}(S)$.
3) $\mathrm{msg}(S) \subseteq \langle a,b\rangle$, $a \in \mathrm{msg}(S)$ and $b \in \mathrm{PF}(S)$.

Algorithm:

INPUT: A numerical semigroup S such that $S \neq \mathbb{N}$.
OUTPUT: The set
$\{\Delta \mid \Delta$ is a numerical semigroup, $\mathrm{e}(\Delta) = 2$ and S is an $\mathrm{I}(\Delta)$-semigroup$\}$.

1) Compute $\mathrm{msg}(S)$ and $\mathrm{PF}(S)$.
2) $A = \{X \subseteq \mathrm{PF}(S) \backslash \{1\} \mid \#X = 2$ and $\mathrm{msg}(S) \subseteq \langle X \rangle\}$.
3) $B = \{\{a, b\} \mid a \in \mathrm{msg}(S), b \in \mathrm{PF}(S) \backslash \{1\}$ and $\mathrm{msg}(S) \subseteq \langle a, b \rangle\}$.
4) $C = \begin{cases} \mathrm{msg}(S) & \text{if } \mathrm{e}(S) = 2 \\ \emptyset & \text{otherwise.} \end{cases}$
5) Return $\{\langle X \rangle \mid X \in A \cup B \cup C\}$.

Example:

$S = \langle 4, 5, 6, 7 \rangle$.

1) $\mathrm{msg}(S) = \{4, 5, 6, 7\}$ and $\mathrm{PF}(S) = \{1, 2, 3\}$.
2) $A = \{\{2, 3\}\}$.
3) $B = \{\{2, 5\}\}$.
4) $C = \emptyset$.
5) $\{\Delta \mid \Delta$ is a numerical semigroup, $\mathrm{e}(\Delta) = 2$ and S is an $\mathrm{I}(\Delta)$-semigroup$\} = \{\langle 2, 3\rangle, \langle 2, 5\rangle\}$.

Example:

$S = \langle 5, 7, 9 \rangle$.

1) $\mathrm{msg}(S) = \{5, 7, 9\}$ and $\mathrm{PF}(S) = \{11, 13\}$.
2) $A = \emptyset$.
3) $B = \emptyset$.
4) $C = \emptyset$.
5) $\{\Delta \mid \Delta$ is a numerical semigroup, $\mathrm{e}(\Delta) = 2$ and S is an $\mathrm{I}(\Delta)$-semigroup$\} = \emptyset$.

Principal $\mathrm{I}(\langle a, b \rangle)$-semigroups

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Proposition: Let S be a numerical semigroup, $x \in S \backslash \{0, 1\}$ and $T = (\{x\} + S) \cup \{0\}$. Then $\mathrm{m}(T) = x$, $\mathrm{F}(T) = \mathrm{F}(S) + x$ and $\mathrm{g}(T) = \mathrm{g}(S) + x - 1$.

$$\begin{array}{rccc} f: & \mathbb{N}\times\{0,1,\cdots,a-1\} & \longrightarrow & \langle a,b\rangle \\ & (\lambda,\mu) & \longmapsto & \lambda a+\mu b \end{array}$$

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If $(\lambda, \mu) \in \mathbb{N} \times \{0, 1, \cdots, a-1\}$, $(\lambda, \mu) \neq (0, 0)$,

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Theorem:

1) $\mathrm{F}(T)=ab+(\lambda-1)a+(\mu-1)b.$
2) $\mathrm{g}(T)=\dfrac{(a-1)(b-1)}{2}+\lambda a+\mu b-1.$
3) $\mathrm{m}(T)=\lambda a+\mu b.$
4) $\mathrm{e}(T)=\lambda a+\mu b.$
5) $\mathrm{msg}(T)=\{(1+\alpha)a+(\mu+\beta)b \mid (\alpha,\beta)\in\{0,1,\cdots,\lambda+b-1\}\times\{0,1,\cdots,\mu-1\}\cup\{0,1,\cdots,\lambda-1\}\times\{\mu,\mu+1,\cdots,a-1\}\}.$

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T is a MED-semigroup

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1. $\mathrm{F}(T) = 40$,
2. $\mathrm{g}(T) = 28$,
3. $\mathrm{m}(T) = 17 = \mathrm{e}(T)$,
4. $\mathrm{msg}(T) = \{(2+\alpha)5 + (1+\beta)7 \mid (\alpha, \beta) \in \{0,1,2,3,4,5,6,7,8\} \times \{0\} \cup \{0,1\} \times \{1,2,3,4\}\} = \{17, 22, 27, 32, 37, 42, 47, 52, 57, 24, 31, 38, 45, 29, 36, 43, 50\}$.

I($\langle a, b \rangle$)-semigroups in general

$\mathrm{I}(\langle a, b\rangle)$-semigroups in general

$T = (\{n_1, \cdots, n_p\} + \langle a, b\rangle) \cup \{0\}$, $\{n_1, \cdots, n_p\}$ a $\langle a, b\rangle$-incomparable set and $p \in \mathbb{N} \backslash \{0, 1\}$.

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Proposition: The following statements are equivalent.

1) X is a $\langle a, b\rangle$-incomparable set with cardinality p.

2) $X = \{\lambda_1 a + \mu_1 b, \lambda_2 a + \mu_2 b, \cdots, \lambda_p a + \mu_p b\}$ where $\{\lambda_1, \mu_1, \cdots, \lambda_p, \mu_p\} \subseteq \mathbb{N}$, $\mu_1 < \mu_2 < \cdots < \mu_p < a$ and $\lambda_p < \cdots < \lambda_2 < \lambda_1 < \lambda_p + b$.

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$(\lambda_1, \mu_1), (\lambda_2, \mu_2), \cdots, (\lambda_p, \mu_p) \in \mathbb{N}^2$, $\mu_1 < \mu_2 < \cdots < \mu_p < a$ and $\lambda_p < \cdots < \lambda_2 < \lambda_1 < \lambda_p + b$.

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$(\lambda_1, \mu_1), (\lambda_2, \mu_2), \cdots, (\lambda_p, \mu_p) \in \mathbb{N}^2$, $\mu_1 < \mu_2 < \cdots < \mu_p < a$ and $\lambda_p < \cdots < \lambda_2 < \lambda_1 < \lambda_p + b$.

Theorem:
$T = (\{\lambda_1 a + \mu_1 b, \lambda_2 a + \mu_2 b, \cdots, \lambda_p a + \mu_p b\} + \langle a, b\rangle) \cup \{0\}$ is an $\mathrm{I}(\langle a, b\rangle)$-semigroup and $\dim_{\langle a,b\rangle}(T) = p$. Moreover, T is an $\mathrm{I}(\langle a, b\rangle)$-semigroup and $\dim_{\langle a,b\rangle}(T) = p$, then T has this form.

Theorem: The following condition holds:

1) $\mathrm{m}(T) = \min\{\lambda_1 a + \mu_1 b, \cdots, \lambda_p a + \mu_p b\}$.
2) $\mathrm{g}(T) = \dfrac{(a-1)(b-1)}{2} + (\lambda_p + b)\mu_1 + \lambda_1(\mu_2 - \mu_1) + \cdots + \lambda_{p-1}(\mu_p - \mu_{p-1}) + \lambda_p(a - \mu_p) - 1$.
3) $\mathrm{F}(T) = \max\{(\lambda_p + b - 1)a + (\mu_1 - 1)b, (\lambda_1 - 1)a + (\mu_2 - 1)b, \cdots, (\lambda_{p-1} - 1)a + (\mu_p - 1)b, (\lambda_p - 1)a + (a-1)b\}$.

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3) $\mathrm{F}(T) = \max\{(\lambda_p + b - 1)a + (\mu_1 - 1)b, (\lambda_1 - 1)a + (\mu_2 - 1)b, \cdots, (\lambda_{p-1} - 1)a + (\mu_p - 1)b, (\lambda_p - 1)a + (a-1)b\}$.

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3) $\mathrm{F}(T) = \max\{(\lambda_p + b - 1)a + (\mu_1 - 1)b, (\lambda_1 - 1)a + (\mu_2 - 1)b, \cdots, (\lambda_{p-1} - 1)a + (\mu_p - 1)b, (\lambda_p - 1)a + (a-1)b\}$.

Example: $a = 5$, $b = 7$, $0 = \mu_1 < \mu_2 = 1 < \mu_3 = 2$ and $1 = \lambda_3 < \lambda_2 = 2 < \lambda_1 = 3$.

Theorem: The following condition holds:

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Example: $a = 5,\ b = 7,\ 0 = \mu_1 < \mu_2 = 1 < \mu_3 = 2$ and $1 = \lambda_3 < \lambda_2 = 2 < \lambda_1 = 3$.
$T = (\{15, 17, 19\} + \langle 5, 7 \rangle) \cup \{0\}$.

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1. $\mathrm{m}(T) = 15$,
2. $\mathrm{g}(T) = \frac{4 \cdot 6}{2} + (1+7)0 + 3(1-0) + 2(2-1) + 1(5-2) - 1 = 19$
3. $\mathrm{F}(T) = \max\{7 \cdot 5 - 7, (3-1)5 + 0 \cdot 7, (2-1)5 + (2-1)7, (1-1)5 + 4 \cdot 7\} = \max\{28, 10, 12, 28\} = 28$.

Algorithm:

INPUT: A Δ-incomparable set, $\{x_1 < \cdots < x_p\}$.
OUTPUT: $\mathrm{msg}(T)$ where $T = (\{x_1, \cdots, x_p\} + \langle a, b\rangle) \cup \{0\}$.

1) Compute $\mathrm{Ap}(\langle a, b\rangle, x_1) \cup \{0\}$.
2) Compute $B = \mathrm{Ap}(\langle a, b\rangle, x_1) \cap \cdots \cap \mathrm{Ap}(\langle a, b\rangle, x_p) = \{w \in \mathrm{Ap}(\langle a, b\rangle, x_1) \mid \{w - x_2, \cdots, x - x_p\} \cap \mathrm{Ap}(\langle a, b\rangle, x_1) = \emptyset\}$.
3) Compute $\mathrm{Ap}(\langle a, b\rangle, 2x_1) = \mathrm{Ap}(\langle a, b\rangle, x_1) + \{0, x_1\}$,
4) Compute $C = \{x_1, \cdots, x_p\} + \{x_1, \cdots, x_p\}$.
5) Compute $D = \bigcap_{\{i,j\} \subseteq \{1,\cdots,p\}} \mathrm{Ap}(\langle a, b\rangle, x_i + x_j) = \{w \in \mathrm{Ap}(\langle a, b\rangle, 2x_1) \mid \{w - c \mid c \in C\} \cap \mathrm{Ap}(\langle a, b\rangle, 2x_1) = \emptyset\}$.
6) Return $(B + \{x_1, \cdots, x_p\}) \cap D$.

$\mathrm{I}(\langle a, b \rangle)$-semigroups with ideal dimension two

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(λ_1, μ_1) and $(\lambda_2, \mu_2) \in \mathbb{N}^2$ such that $\mu_1 < \mu_2 < a$ and $\lambda_2 < \lambda_1 < \lambda_2 + b$.

I($\langle a, b\rangle$)-semigroups with ideal dimension two

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3) $\mathrm{F}(T) = \begin{cases} (\lambda_2 + b - 1)a + (\mu_1 - 1)b & \text{if } ab \geq (\lambda_1 - \lambda_2)a + (\mu_2 - \mu_1)b \\ \text{or} & \\ (\lambda_1 - 1)a + (\mu_2 - 1)b & \text{otherwise.} \end{cases}$

Compute all the $\mathrm{I}(\langle a, b \rangle)$-semigroups with ideal dimension two with

- a fixed genus or
- a fixed Frobenius number.

Theorem: Let $(\lambda, \mu, n) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\} \times \langle a, b \rangle$. Then $T(\lambda, \mu, n) = (\{\lambda a + n, \mu b + n\} + \langle a, b \rangle) \cup \{0\}$ is an $\mathrm{I}(\langle a, b \rangle)$-semigroup and $\dim_{\langle a,b \rangle}(T(\lambda, \mu, n)) = 2$. Moreover, every $\mathrm{I}(\langle a, b \rangle)$-semigroup with ideal dimension two has this form.

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Proposition: If $(\lambda, \mu, n) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\} \times \langle a, b \rangle$, then

1) $\mathrm{m}(T(\lambda, \mu, n)) = \min \{\lambda a + n, \mu b + n\}$.
2) $\mathrm{g}(T(\lambda, \mu, n)) = \dfrac{(a-1)(b-1)}{2} + \lambda \mu - 1 + n$.
3) $\mathrm{F}(T(\lambda, \mu, n)) = \max\{ab - a - b, (\lambda - 1)a + (\mu - 1)b\} + n$.

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$T(\lambda,\mu,n)=T(2,3,17)=(\{27,38\}+\langle 5,7\rangle)\cup\{0\}=$ $\{0,27,32,34,37,38,39,41,\longrightarrow\}$ is an $\mathrm{I}(\langle 5,7\rangle)$-semigroup with ideal dimension two and genus 34.

Algorithm:

INPUT: An integer g such that $g > \dfrac{(a-1)(b-1)}{2}$.

OUTPUT:

$\{S \mid S$ is an $\mathrm{I}(\langle a, b\rangle)$-semigroup, $\dim_{\langle a,b\rangle}(S) = 2$ and $\mathrm{g}(S) = g\}$.

1) Compute $A = \{(\lambda, \mu, n) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\} \times \langle a, b\rangle \mid \lambda \cdot \mu + n = g + 1 - \frac{(a-1)(b-1)}{2}\}$.
2) Return $\{T(\lambda, \mu, n) \mid (\lambda, \mu, n) \in A\}$.

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$(\lambda, \mu, n) \in \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4\} \times \langle 5, 7\rangle$, verifying one of the following conditions:

1) $(\lambda - 1)5 + (\mu - 1)7 < 23$ and $n = 20$.
2) $(\lambda - 1)5 + (\mu - 1)7 > 23$ and $(\lambda - 1)5 + (\mu - 1)7 + n = 43$.

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Then

$(\lambda,\mu,n)=(2,2,20)$, then $T(\lambda,\mu,n)=T(2,2,20)=$ $(\{30,34\}+\langle 5,7\rangle)\cup\{0\}=\{0,30,34,35,37,39,40,41,42,44\longrightarrow\}$ is an $\mathrm{I}(\langle 5,7\rangle)$-semigroup with ideal dimension two and Frobenius number 43.

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$(\lambda,\mu,n)=(4,3,14)$, then $T(\lambda,\mu,n)=T(4,3,14)=(\{34,35\}+\langle 5,7\rangle)\cup\{0\}=\{0,34,35,39,40,41,42,44,\longrightarrow\}$ is an $\mathrm{I}(\langle 5,7\rangle)$-semigroup with ideal dimension two and Frobenius number 43.

Algorithm:

INPUT: An integer F such that $F \geq ab - a - b$.

OUTPUT:

$\{S \mid S$ is an $\mathrm{I}(\langle a, b\rangle) -$ semigroup, $\dim_{\langle a,b\rangle}(S) = 2$ and $\mathrm{F}(S) = F\}$.

1) If $F - (ab - a - b) \notin \langle a, b\rangle$, then $A = \emptyset$.
2) If $F - (ab - a - b) \in \langle a, b\rangle$, then $A = \{(\lambda, \mu, F - (ab - a - b)) \mid (\lambda, \mu) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\}$ and $(\lambda - 1)a + (\mu - 1)b < ab - a - b\}$.
3) $B = \{(\lambda, \mu, n) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\} \times \langle a, b\rangle \mid (\lambda-1)a+(\mu-1)b > ab-a-b$ and $n = F-(\lambda-1)a-(\mu-1)b\}$.
4) Return $\{T(\lambda, \mu, n) \mid (\lambda, \mu, n) \in A \cup B\}$.

Thanks for your attention!!