On the isomorphism problem for monoids of product-one sequences

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Outline

Product-one sequences

Let G be a group.

- An element of the free abelian monoid $\mathcal{F}(G)$ with a basis G is said to be a sequence over G, i.e., every sequence S over G has the form

$$S = (g_1, g_2, \ldots, g_\ell) = g_1 \cdot g_2 \cdot \ldots \cdot g_\ell = \prod\nolimits^{\bullet}_{g \in G} g^{[\mathsf{v}_g(S)]} .$$

- S is called a product-one sequence if the terms can be ordered such that their product (in G) is equal to the identity element of G.

ex) If $G = \{\pm E, \pm I, \pm J, \pm K\}$ is the quaternion group of order 8, then a sequence

$$I^{[4]} \cdot J^{[2]} = I \cdot I \cdot I \cdot I \cdot J \cdot J$$

is product-one, because $E = IIIJIJ$

Product-one sequences

- The set $\mathcal{B}(G)$ of all product-one sequences is a submonoid of $\mathcal{F}(G)$, and it is called the monoid of product-one sequences over G.
- An atom (or irreducible element) in $\mathcal{B}(G)$ is called a minimal product-one sequence.
- The Davenport constant $\mathsf{D}(G)$ is the maximal length of an atom in $\mathcal{B}(G)$.

$\rightsquigarrow$ While earlier work often focussed on the case of abelian groups, sequences over non-abelian groups have received wide attention due to their applications in various branches of algebra, such as invariant theory and factorization theory.

Factorizations and Set of lengths

Let H be a monoid, that is, a commutative, cancellative semigroup with identity.

Q. Are the arithmetical properties of two objects H_1 and H_2 characteristic for H_1 and H_2?

$\rightsquigarrow$ The sets of lengths are the best investigated properties.

- If $a = u_1 \cdot \ldots \cdot u_k$ for atoms $u_1, \ldots, u_k$ in H, k is called the length of factorization of a, and we denote by

$$\mathsf{L}(a) = \{k \in \mathbb{N} \mid a \text{ has a factorization of length } k\}.$$

- $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ denotes the system of sets of lengths of H.

ex) Let K be an algebraic number field with class group G. Then there exists a factorization preserving map β from $\mathcal{O}_K$ to the monoid of product-one sequences over the class group G of K. More precisely, $\beta(a) = [P_1] \bullet \ldots \bullet [P_k]$, where $a\mathcal{O}_K = P_1 \cdots P_k$ is the factorization into prime ideals.

The Characterization Problem

- Characterization Problem
  Given two finite (abelian) groups G_1 and G_2 such that $\mathcal{L}(\mathcal{B}(G_1)) = \mathcal{L}(\mathcal{B}(G_2))$, does it follow that $G_1 \cong G_2$?

It holds true so far for the following groups:

- Geroldinger, Schmid, Zhong
  - G is an elementary 2-groups.
  - $G \cong C_{n_1} \oplus C_{n_2}$, where $n_1, n_2 \in \mathbb{N}$ with $n_1 \mid n_2$ and $n_1 + n_2 > 4$.
  - $G \cong C_n^r$, where $r, n \in \mathbb{N}$ with $r \leq n - 3$.

- Geroldinger, Grynkiewicz, OH, Zhong
  - G is a finite group with $\mathsf{D}(G) \leq 6$.
  - $G \cong D_{2n}$ with n odd.

The Isomorphism Problem

- Isomorphism Problem
  Given two finite groups G_1 and G_2 such that $\mathcal{B}(G_1) \cong \mathcal{B}(G_2)$, does it follow that $G_1 \cong G_2$?

$\leadsto$ An affirmative answer to the Isomorphism Problem is a necessary condition for an affirmative answer to the Characterization Problem.

- The answer to the Isomorphism Problem was known so far only for abelian groups, and its proof heavily depends on the ideal-theoretic properties of monoids.

Outline

Krull monoids

- A monoid homomorphism $\varphi \colon H \to D$ is a divisor theory if the following conditions hold;
  1. For $a, b \in H$, $a \mid b$ in H if and only if $\varphi(a) \mid \varphi(b)$ in D,
  2. $D = \mathcal{F}(P)$ is a free abelian monoid,
  3. For $p \in P$, there exist $a_1, \ldots, a_n \in H$ such that $p = \gcd(\varphi(a_1), \ldots, \varphi(a_n))$.

⇝ The main consequence of divisor theories is that it has a universal property.

- The complete integral closure of a monoid H is $\widehat{H} = \{x \in \mathsf{q}(H) \mid \exists\, c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N}\}$.

Krull monoids

A monoid H is Krull if the following equivalent conditions holds;

(a) H satisfies the ACC on v-ideals, and $H = \widehat{H}$.

(b) H satisfies the ACC on v-ideals, and every non-empty v-ideal of H is v-invertible.

(c) The map $H \to \mathcal{I}_v^*(H)$, given by $a \mapsto aH$, is a divisor theory.

(d) H has a divisor theory.

ex) The ring $\mathcal{O}_K$ of algebraic integers is an 1-dimensional Krull domain.

$\rightsquigarrow$ Every Krull monoid satisfies the transfer machinery.

The Isomorphism Problem: Abelian Case

- The followings are equivalent;
  (a) G is abelian.
  (b) $\mathcal{B}(G)$ is Krull.
  (c) $\mathcal{B}(G)$ is transfer Krull.
  (d) $\mathcal{B}(G) \hookrightarrow \mathcal{F}(G)$ is a divisor theory.

- The map

$$\mathcal{F}(G)/\mathcal{B}(G) \to G\,, \quad (g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell)\mathsf{q}(\mathcal{B}(G)) \mapsto g_1 \cdots g_\ell$$

  is a group isomorphism.

- By the Uniqueness Theorem for divisor theories, $\mathcal{B}(G_1) \cong \mathcal{B}(G_2)$ implies that $\mathcal{F}(G_1) \cong \mathcal{F}(G_2)$, so that

$$G_1 \cong \mathcal{F}(G_1)/\mathcal{B}(G_1) \cong \mathcal{F}(G_2)/\mathcal{B}(G_2) \cong G_2 .$$

Outline

1. The Isomorphism Problem
2. Abelian Case
3. Result

The Isomorphism Problem: General Case

Let G be a group and G' be the commutator subgroup.

- For $S = g_1 \cdot \ldots \cdot g_\ell \in \mathcal{F}(G)$,

$$\pi(S) = \{g_{\sigma(1)} \cdots g_{\sigma(\ell)} \mid \sigma \text{ is a permutation of } [1, \ell]\} .$$

- Geroldinger-Grynkiewicz-OH-Zhong, 2022
  If G is finite, then $\widehat{\mathcal{B}(G)} = \{S \in \mathcal{F}(G) \mid \pi(S) \subseteq G'\}$ is Krull.

Fadinger-Zhong consider the following monoid:

- $\mathcal{B}(G)^\star := \{S \in \mathcal{F}(G) \mid \pi(S) \subseteq G'\} \subseteq \mathcal{F}(G)$ is a submonoid with

$$\mathcal{B}(G) \subseteq \mathcal{B}(G)^\star \subseteq \mathcal{F}(G) .$$

- G is abelian $\implies$ $\mathcal{B}(G) = \mathcal{B}(G)^\star$.
- G is perfect $\implies$ $\mathcal{B}(G)^\star = \mathcal{F}(G)$.

The Isomorphism Problem: General Case

- Fadinger-Zhong, 2023
  1. The map
  $$\begin{array}{rcll} \mathcal{F}(G)/\mathcal{B}(G) & \to & G/G' & \\ S\mathsf{q}(\mathcal{B}(G)) & \mapsto & gG' & \text{for } g \in \pi(S) \end{array}$$
  is a group isomorphism.
  2. $\mathcal{B}(G)^\star$ is a Krull monoid with $\mathcal{B}(G) \subseteq \widehat{\mathcal{B}(G)} \subseteq \mathcal{B}(G)^\star$ and $\mathcal{B}(G)^\star \hookrightarrow \mathcal{F}(G)$ is a divisor theory.
  3. $\widehat{\mathcal{B}(G)}$ is Krull if and only if $\widehat{\mathcal{B}(G)} = \mathcal{B}(G)^\star$.
  4. If G is torsion, then $\widehat{\mathcal{B}(G)} = \mathcal{B}(G)^\star$.

For groups G_1 and G_2,

$$\begin{array}{rcl} \mathcal{B}(G_1) \cong \mathcal{B}(G_2) & \Longrightarrow & \widehat{\mathcal{B}(G_1)} \cong \widehat{\mathcal{B}(G_2)} \\ & \overset{??}{\Longrightarrow} & \mathcal{B}(G_1)^\star \cong \mathcal{B}(G_2)^\star \end{array}$$

Theorem (Geroldinger-OH)

Let G_1 and G_2 be groups and suppose that G_1 is a torsion group. Then, $\mathcal{B}(G_1) \cong \mathcal{B}(G_2)$ if and only if $G_1 \cong G_2$.

Ingredient

- The *opposite group* G^{op} of a group G has the same underlying set and its group operation is defined by $g_1 \cdot^{\mathsf{op}} g_2 := g_2 g_1$ for all $g_1, g_2 \in G$.
- The map $\psi \colon G \to G^{\mathsf{op}}$, defined by $\psi(g) = g^{-1}$ for all $g \in G$, is a group isomorphism.
- A group homomorphism $\varphi \colon G_1 \to G_2$ is an *anti-homomorphism* if $\varphi(g_1 g_2) = \varphi(g_2)\varphi(g_1)$ for all $g_1, g_2 \in G_1$.

Sketch of the Proof

Suppose that $\mathcal{B}(G_1) \cong \mathcal{B}(G_2)$.

- $\widehat{\mathcal{B}(G_1)} \cong \widehat{\mathcal{B}(G_2)}$, and for each i, $\widehat{\mathcal{B}(G_i)} \hookrightarrow \mathcal{F}(G_i)$ is a divisor theory.
- $\mathcal{F}(G_1) \cong \mathcal{F}(G_2)$ (by the uniqueness for divisor theories).
- We have a bijection $\varphi\colon G_1 \to G_2$ such that
  1. $\operatorname{ord}(g) = \operatorname{ord}\big(\varphi(g)\big)$ for all $g \in G_1$,
  2. $\varphi(1_{G_1}) = 1_{G_2}$,
  3. $S = g_1 \cdot \ldots \cdot g_\ell \in \mathcal{B}(G_1)$ iff $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_\ell) \in \mathcal{B}(G_2)$,
  4. $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G_1$, and
  5. for all $g_1, g_2 \in G_1$, we have

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) \quad \text{or} \quad \varphi(g_1 g_2) = \varphi(g_2)\varphi(g_1)\,.$$

- $\varphi\colon G_1 \to G_2$ is either a group isomorphism or a group anti-isomorphism.

References

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Thank you for your attention!