International Meeting of Numerical Semigroups

Ideals of affine semigroups, $I(S)$-semigroups and MED-semigroups

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Background

$(S, +)$

$S \subseteq \mathbb{N}$

$0 \in S$

} Monoid

$\mathbb{N} \setminus S$ is finite

} Numerical semigroup

Background

$(S, +)$

- $S \subseteq \mathbb{N}^p$ $(\mathcal{C})$
- $0 \in S$
- S is finitely generated → $msg(S)$
- ~~$\mathbb{N} \setminus S$~~ $\mathcal{C} \setminus S$ is finite

~~Monoid~~ Affine semigroup

~~Numerical~~ $\mathcal{C}$-semigroup

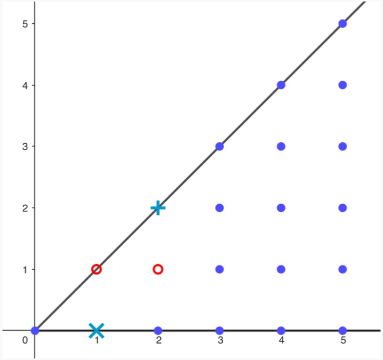
$\mathcal{C}$ is the minimal positive integer cone containing S.

Background: Invariants

Let S be a $\mathcal{C}$-semigroup, and let $\preceq$ be a monomial order.

$\mathcal{H}(S) = \mathcal{C} \setminus S$
$g(S) = \#\mathcal{H}(S)$
$\mathrm{Fb}(S) = \max_{\preceq} \mathcal{H}(S)$

$\{\tau_1, \tau_2, \ldots, \tau_t\}$ are the extremal rays of $\mathcal{C}$
$\mathrm{mult}_i(S) = \min_{\leq}(\tau_i \cap S)$, $1 \leq i \leq t$



Ideals of an affine semigroup

Definition: Any subset P of an affine semigroup S is an **ideal** of S if

$$P + S \subseteq P.$$

Example

Any subset A of an affine semigroup S.

$$P = A + S \text{ is an ideal of } S.$$

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It could exist two <u>different</u> finite subsets $A_1, A_2 \subseteq S$ such that $A_1 + S = A_2 + S$.

Ideals of an affine semigroup

Theorem:

Let S be an affine semigroup. Then,

$\{a \in S \mid a - a' \notin S, \forall a' \in S \setminus \{a\}\}$

$$\{A + S \mid A \text{ is } S\text{-incomparable}\}$$

is the set formed by all the ideals of S. Moreover, if A_1 and A_2 are different S-incomparable sets, then $A_1 + S \neq A_2 + S$.

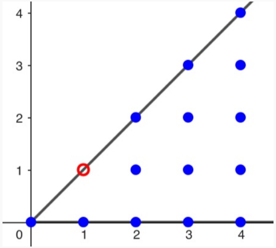
Ideals of an affine semigroup

Lemma:

Given a $\mathcal{C}$-semigroup S, and P an ideal of S. Then

$$P \cup \{0\} \text{ is a } \mathcal{C}\text{-semigroup} \Leftrightarrow \begin{array}{l} \exists Y \subset P \setminus \{0\} \text{such that} \\ Y \cap \tau_i \neq \emptyset \; \forall 1 \leq i \leq t. \end{array}$$

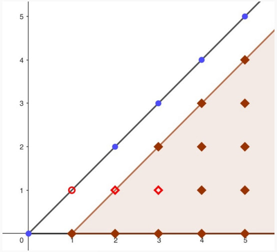
$S = \mathcal{C} \setminus \{(1,1)\}$



$\bullet \in S$

$\circ \in \mathcal{H}(S)$

$P = \{(1,0)\} + S$



$\blacklozenge \in S$

$\diamond \in \mathcal{H}(S)$

$I(S)$-semigroups

$$P \text{ is an ideal of } S \begin{cases} P \cup \{0\} \text{ is not a } \mathcal{C} \text{ semigroup} \\ \underbrace{P \cup \{0\}}_{T} \text{ is a } \mathcal{C}\text{-semigroup} \end{cases}$$

Definition

Let S be a $\mathcal{C}$-semigroup. A $\mathcal{C}$-semigroup T is an $I(S)$-semigroup of S if $T \setminus \{0\}$ is an ideal of S.

$I(S)$-semigroups

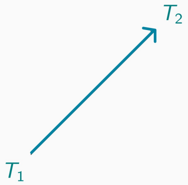
$\mathcal{J}(\mathcal{S}) := \{T \mid T \text{ is an } I(S)\text{-semigroup}\}$.

We define the tree $G(\mathcal{J}(\mathcal{S}))$:

- The vertex set is $\mathcal{J}(\mathcal{S})$
- $(T_1, T_2) \in \mathcal{J}(\mathcal{S}) \times \mathcal{J}(\mathcal{S})$ is an edge if $T_2 = T_1 \cup \{\mathcal{O}_S(T_1)\}$.

$\mathcal{O}_A(B) = \max_{\preceq}(A \setminus B)$.

If (T_1, T_2) is an edge, we say that T_1 is a child of T_2.



$I(S)$-semigroups

Lemma:
Let S be a $\mathcal{C}$-semigroup and T be a non-proper $I(S)$-semigroup. Then, $T \cup \{\mathcal{O}_S(T)\} \in \mathcal{J}(\mathcal{S})$.

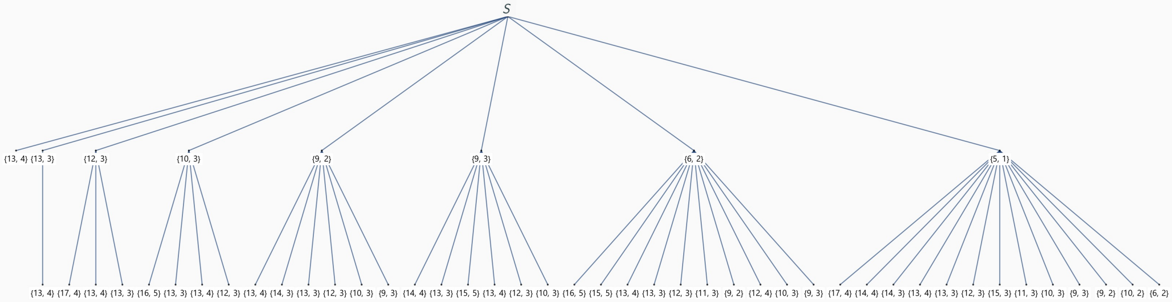
Theorem:
For any $\mathcal{C}$-semigroup S, $G(\mathcal{J}(\mathcal{S}))$ is a tree with root S. Furthermore, the set of children of any $T \in \mathcal{J}(\mathcal{S})$ is the set

$$\{T \setminus \{x\} \mid x \in Minimals_{\leq_S}(msg(T)) \text{ and } x \succ \mathcal{O}_S(T)\}.$$

Let $x, x' \in S$, $x \leq_S x' \Leftrightarrow x' - x \in S$.

Example:

Let S be the $\mathcal{C}$-semigroup such that
$msg(S) = \{(5,1),(6,2),(9,2),(9,3),(10,3),(12,3),(13,4),(13,3)\}$.



$I(S)$-semigroups

Given $f \in \mathcal{C}$ and a monomial order $\preceq$, consider
$A_S(f) = \{x \in S \mid x \prec f \text{ and } f - x \notin S\}$.

Proposition:
Let S be a $\mathcal{C}$-semigroup, $f \in \mathcal{C}$ greater than or equal to $\mathrm{Fb}(S)$, and $\preceq$ a monomial order. The following conditions are equivalent:

T is an $I(S)$-semigroup with Frobenius element $\mathrm{Fb}(T) = f$.

$$\Updownarrow$$

$T = X \sqcup \{x \in \mathcal{C} \mid x \succ f\} \cup \{0\}$, where X is a subset of $A_S(f)$ such that if there exists $x \in X$ with $x + s \prec f$ for some $s \in S$, then $x + s \in X$.

$I(S)$-semigroups

Example:

Let S be the $\mathcal{C}$-semigroup mentioned, and consider $f = (11, 3)$. Fixed the degree lexicographic order.

All $I(S)$-semigroups with Frobenius element $f = (11, 3)$ are determined by $X \sqcup \{x \in \mathcal{C} \mid x \succ (11, 3)\} \cup \{0\}$ for each X in the set

$\{\emptyset\}$,
$\{(9, 2)\}$,
$\{(9, 3)\}$,
$\{(10, 2)\}$,
$\{(10, 3)\}$,
$\{(9, 2), (9, 3)\}$,
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$I(S)$-semigroups

Theorem:
Let S be a $\mathcal{C}$-semigroup and $M=\{m_1,\dots,m_t\}\subset S\setminus\{0\}$ such that $m_i\in\tau_i\setminus\{0\}$ for all $i\in[t]$. The set of all $I(S)$-semigroups with i-multiplicity m_i, $1\leq i\leq m$, is

$$\{\big((M\sqcup X)+S\big)\cup\{0\}\mid X\in\mathcal{P}(\mathcal{H}(M+S)\cap S)\}.$$

$I(S)$-semigroups

Example

Again, let S be the $\mathcal{C}$-semigroup mentioned, and consider $M = \{(10,2),(6,2)\}$. There exists 2047 sets X determining all $I(S)$-semigroups with the i-multiplicities in M.

There exists 351 $I(S)$-semigroups different such that M is the union of the i-multiplicities of any of them.

MED-semigroups

Let $S \subset \mathbb{N}^p$ be an affine semigroup with t extremal ray, such that

$$msg(S) = \{ \underbrace{n_1, \ldots, n_t}_{E = \cup_{i\in[t]} \mathrm{mult}_i(S)}, \underbrace{n_{t+1}, \ldots, n_r}_{A} \}.$$

The Apery set of S respect to $m \in S \setminus \{0\}$ is the set

$$Ap(S, m) = \{ s \in S \mid s - m \notin S \}.$$

- For any non-numerical affine semigroup, this set is not finite
- $\cap_{i\in[t]} Ap(S, n_i)$ is finite
- $A \subset \cap_{i\in[t]} Ap(S, n_i)$

MED-semigroups

Definition

Given $S \subset \mathbb{N}^p$ an affine semigroup minimally generated by $E \sqcup A$, S is a maximal embedding dimension affine semigroup (MED-semigroup) if

$$\cap_{i\in[t]} Ap(S, n_i) = A \sqcup \{0\}.$$

MED-semigroups

Definition

Given $S \subset \mathbb{N}^p$ an affine semigroup minimally generated by $E \sqcup A$, S is a maximal embedding dimension affine semigroup (MED-semigroup) if

$$\cap_{i\in[t]} Ap(S, n_i) = A \sqcup \{0\}.$$

Theorem [García-Sánchez, Rosales, 2009]

Let S be a numerical semigroup. Then,

$$T \text{ is a MED-semigroup} \Leftrightarrow T = \{m\} + S, \text{ for some } m \in S \setminus \{0\}.$$

MED-semigroups

Theorem

Let S be an affine semigroup, $M = \{m_1, \ldots, m_t\} \subset S$ such that $M \cap (\tau_i \setminus \{0\}) \neq \emptyset$ for every $i \in [t]$, and the ideal of S defined by $M + S$. Then, the $I(S)$-semigroup $T = (M + S) \cup \{0\}$ is a MED-semigroup.

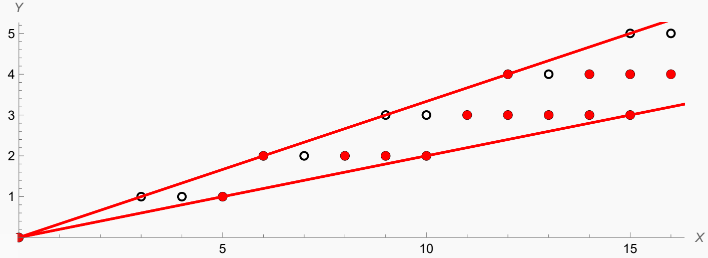
<= ?

MED-semigroups

Example

Let $S \subset \mathbb{N}^2$ be the affine semigroup such that

$$msg(S) = \{(5,1),(6,2),(8,2),(9,2),(12,3)\},$$



$\bullet \in S$

$\circ \in \mathcal{H}(S)$

MED-semigroup generated by $\{(5,1),(6,2),(8,2),(9,2),(12,3)\}$.

Open problem

No additional constructions have been identified. We threw the open problem of whether more distinct constructions exist beyond those already proposed for obtaining MED-semigroups.

Thank you for your attention!

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