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On Monoids of Weighted Zero-Sum Sequences

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Outline

Sequences I

Let G be an additive abelian group, $G_0 \subset G$ be a subset, and $\Gamma \subset \operatorname{End}(G)$ of a subset of the endomorphism group.

- A sequence $S = g_1 \cdot \ldots \cdot g_\ell$ over G_0: finite, unordered sequence of terms from G_0, repetition allowed;
- $|S| = \ell$ denotes its length and $\sigma(S) = g_1 + \ldots + g_\ell$ its sum.
- Sequences are considered as elements of the free abelian monoid $\mathcal{F}(G_0)$.
- $\sigma_\Gamma(S) = \{\gamma_1(g_1) + \ldots + \gamma_\ell(g_\ell) \colon \gamma_1, \ldots, \gamma_\ell \in \Gamma\}$ the set of Γ-weighted sums of S.
- $\Sigma(S) = \{\sigma(T) \colon 1 \ne T \in \mathcal{F}(G),\ T \mid S\}$.
- $\Sigma_\Gamma(S) = \bigcup_{1 \ne T \mid S} \sigma_\Gamma(T)$.

Monoids of Weighted Zero-Sum Sequences

A sequence $S \in \mathcal{F}(G_0)$ is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- *zero-sum free* if $0 \notin \Sigma(S)$,
- a Γ*-weighted zero-sum sequence* if $0 \in \sigma_\Gamma(S)$, and
- Γ*-weighted zero-sum free* if $0 \notin \Sigma_\Gamma(S)$.

We denote by

- $\mathcal{B}(G_0)$ the monoid of zero-sum sequences over G_0,
- $\mathcal{B}_\Gamma(G_0)$ the monoid of Γ-weighted zero-sum sequences over G_0,
- $\mathcal{B}_\pm(G_0)$ the monoid of plus-minus weighted zero-sum sequences over G_0, in case that $\Gamma = \{\mathrm{id}, -\mathrm{id}\}$.

Weighted zero-sums are studied since 2006: Sukumar das Adhikari.

Davenport Constants

- $\mathsf{D}(G) := \mathsf{D}\big(\mathcal{B}(G)\big)$ is the maximal length of an irreducible element in $\mathcal{B}(G)$;
  $\mathsf{d}(G)$ is the maximal length of a zero-sum free sequence.

- $\mathsf{D}_\Gamma(G) := \mathsf{D}\big(\mathcal{B}_\Gamma(G)\big)$ is the maximal length of an irreducible element in $\mathcal{B}_\Gamma(G)$;
  $\mathsf{d}_\Gamma(G)$ is ....

- $\mathsf{D}_\pm(G) := \mathsf{D}\big(\mathcal{B}_\pm(G)\big)$ is the maximal length of an irreducible element in $\mathcal{B}_\pm(G)$;
  $\mathsf{d}_\pm(G)$ is ...

The Davenport constant I

The Davenport constant $\mathsf{D}(G)$ is the maximal length of a minimal zero-sum sequence over G, thus

$$\mathsf{D}(G) = \max\{\ell \mid S = g_1 \cdot \ldots \cdot g_\ell \text{ is an irreducible element of } \mathcal{B}(G)\} .$$

Let

$$G = C_{n_1} \oplus \ldots \oplus C_{n_r} \quad \text{with} \quad 1 < n_1 \mid \ldots \mid n_r$$

- $1 + \sum_{i=1}^{r} (n_i - 1) \le \mathsf{D}(G)$.
- 1960s: Equality holds for p-groups, for $r \le 2$, and for others.
- For every $r \ge 4$ there are infinitely many groups G of rank r for which inequality holds.
- OPEN PROBLEM Determine $\mathsf{D}(G)$ in terms of $(n_1, \ldots, n_r)$.

$$\mathsf{D}(C_n \oplus C_n \oplus C_n) = ?$$

Outline

Davenport Constants II

Let K be an algebraic number field with class group G, ring of integers $\mathcal{O}_K$, and Galois group Γ. If $[K : \mathbb{Q}] = 2$, then $\Gamma = \{\mathrm{id}, \tau\}$, with $\tau(g) = -g$ for $g \in G$, whence $\mathsf{D}_\Gamma(G) = \mathsf{D}_\pm(G)$.

Rogers 1962: $\mathsf{D}(G)$ is the maximal number of prime ideals occurring in the the prime ideal factorization of an irreducible element of $\mathcal{O}_K$.

Halter-Koch 2014:

- $1 + \mathsf{d}(G) = \mathsf{D}(G)$ is the smallest ℓ such that every product of ℓ nonzero ideals of $\mathcal{O}_K$ is contained in a proper principal ideal.
- $1 + \mathsf{d}_\Gamma(G)$ is the smallest $\ell \in \mathbb{N}$ with the following property:
    - If $q_1, \ldots, q_\ell$ are pairwise coprime positive integers such that their product q is the norm of an ideal of $\mathcal{O}_K$, then some divisor $t > 1$ of q is the norm of a principal ideal of $\mathcal{O}_K$.

Davenport Constants III

Halter-Koch 2014:

- Let $\Delta \in \mathbb{Z}$ be not a square, $\Delta \equiv 0$ or $1 \bmod 4$, and let G be the class group of non-negative definite primitive integral binary quadratic forms of discriminant Δ. Then $1 + \mathsf{d}_{\pm}(G)$ is the smallest $\ell \in \mathbb{N}$ with the following property:
    - If $q_1, \ldots, q_\ell$ are pairwise coprime positive integers such that their product q is properly represented by some class of G, then some divisor $t > 1$ of q is represented by the principal class of the discriminant Δ.

WHY?

Sets of Lengths in Monoids

Monoid H: comm., cancellative semigroup with 1_H.

- If $a = u_1 \cdot \ldots \cdot u_k$ where $u_1, \ldots, u_k \in \mathcal{A}(H)$, then k is called the length of the factorization, and
- $\mathsf{L}_H(a) = \{k \mid a \text{ has a factorization of length } k\} \subset \mathbb{N}$ is the length set of a.
- H is half-factorial if $|\mathsf{L}(a)| = 1$ for all $a \in H$.
- $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ is the system of all length sets

If $a = u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_\ell$, then

$$a^N = (u_1 \cdot \ldots \cdot u_k)^i (v_1 \cdot \ldots \cdot v_\ell)^{N-i} \quad \text{for all } i \in [0, N] \,.$$

FACT. Either H is half-factorial or
for every $N \in \mathbb{N}$ there is $a_N \in H$ with $|\mathsf{L}(a_N)| > N$.

Transfer Homomorphisms

Halter-Koch 1997:

A monoid homomorphism $\theta\colon H \to B$ is a transfer homomorphism if

- θ is surjective up to units; only units are mapped onto units.
- θ allows to lift factorizations: if $\theta(a) = BC$, then there are $b, c \in H$ such that $\theta(b) \simeq B$, $\theta(c) \simeq C$, and $a = bc$.

Philosophy: H is the object of interest and B is simpler than H.

Transfer homomorphisms allow to pull back arithmetic properties from B to H. In particular,

- $\mathsf{L}_H(a) = \mathsf{L}_B\big(\theta(a)\big)$ for all $a \in H$.
- $\mathcal{L}(H) = \mathcal{L}(B)$.

Divisor Theories I

Concept of divisor theory: origins of alg. number theory, Borevic-Safarevic, Clifford, Skula, Gundlach, Halter-Koch

A monoid homomorphism $\varphi\colon H \to D$ is said to be a

- divisor homomorphism if, for all $a, b \in H$,

$$a \mid b \text{ in } H \quad \text{if and only if} \quad \varphi(a) \mid \varphi(b) \text{ in } D\,.$$

- divisor theory if
  - φ is a divisor homomorphism,
  - $D = \mathcal{F}(P)$ is free abelian,
  - For every $p \in P$, there are $a_1, \ldots, a_m \in H$ such that $p = \gcd\big(\varphi(a_1), \ldots, \varphi(a_m)\big)$.

Divisor Theories II: Classic Example

Let K be an algebraic number field and $\mathcal{O}_K$ its ring of integers.

Then

- $\mathcal{O}_K^\bullet = (\mathcal{O}_K \setminus \{0\})$ is a monoid.
- Since $\mathcal{O}_K$ is Dedekind,

$$\varphi\colon \begin{cases} \mathcal{O}_K^\bullet & \to \mathcal{I}^*(\mathcal{O}_K) = \mathcal{F}(\mathrm{spec}^\bullet(\mathcal{O}_K)) \\ a & \mapsto a\mathcal{O}_K \end{cases}$$

is a divisor theory.

A transfer homomorphism from $\mathcal{O}_K$ to a monoid of zero-sum sequences

Let $\varphi\colon \mathcal{O}_K^\bullet \to \mathcal{I}^*(\mathcal{O}_K)$, $\varphi(a) = a\mathcal{O}_K$ for all $a \in \mathcal{O}_K^\bullet$

$$\begin{array}{ccc} \mathcal{O}_K^\bullet & \longrightarrow & \mathcal{I}^*(\mathcal{O}_K) \\ \beta \big\downarrow & & \big\downarrow \widetilde{\beta} \\ \mathcal{B}(G) & \longrightarrow & \mathcal{F}(G) \end{array}$$

Let $\widetilde{\beta}$ map ideals to the sequence of ideal classes:

$$I = P_1 \cdot \ldots \cdot P_\ell \in \mathcal{I}^*(\mathcal{O}_K) \quad \text{to} \quad \beta(I) = [P_1] \cdot \ldots \cdot [P_\ell] \in \mathcal{F}(G)$$

and, by definition of the class group, we have

- I is a principal ideal $\iff$ $\beta(I)$ is a zero-sum sequence.
- β is a transfer homomorphism.

Krull Monoids

A monoid is Krull if the following equivalent conditions hold.

(a) *H satisfies the ACC on v-ideals and is completely integrally closed.*

(b) *H satisfies the ACC on v-ideals and every non-empty v-ideal is v-invertible.*

(c) *The map $\partial \colon H \to \mathcal{I}_v^*(H)$ is a divisor theory.*

(d) *H has a divisor theory.*

(e) *There is a divisor homomorphism $\varphi \colon H \to \mathcal{F}(P)$.*

FACT. Every Krull monoid allows a transfer homomorphism to a monoid of zero-sum sequences.

Transfer homomorphisms to monoids of weighted zero-sum sequences I

Let K be a Galois algebraic number field with

- ring of integers $\mathcal{O}_K$, class group G, Galois group Γ,
- $\mathsf{N} : \mathcal{I}^*(\mathcal{O}_K) \to \mathbb{N}$ the absolute norm, and
- the norm monoid $\mathsf{N}(\mathcal{H}_K) = \{\mathsf{N}(a\mathcal{O}_K) \colon a \in \mathcal{O}_K^\bullet\}$.

(Schmid et al.) There is a transfer homomorphism

$$\theta \colon \mathsf{N}(\mathcal{H}_K) \to \mathcal{B}_\Gamma(G)\,.$$

Transfer homomorphisms to monoids of weighted zero-sum sequences II

Similar transfer results exist for

- Galois invariant orders in algebraic number fields.
- Monoids of totally positive elements in Galois invariant orders.
- Monoids of elements representable by certain binary quadratic forms
- Norm monoids: recently studied by Coykendall+Hasenauer).

See the References.

Outline

Finitely generated and ACCs

FACT. $\mathcal{B}(G)$ is Krull and
it is finitely generated if and only if G is finite.

Theorem

1. *$\mathcal{B}_{\pm}(G)$ is finitely generated if and only if G is finite.*
2. *$\mathcal{B}_{\pm}(G)$ satisfies the ACC on v-ideals if and only if G is the direct sum of an elem. 2-group and a finite group.*
3. *The following are equivalent.*
   - *$\mathcal{B}_{\pm}(G)$ is Krull.*
   - *$\mathcal{B}_{\pm}(G)$ is completely integrally closed*
   - *$\mathcal{B}(G)$ allows a transfer hom. to a Krull monoid.*
   - *G is an elementary 2-group.*

The Isomorphism Problem I

Theorem

Let G and G' be abelian groups.
Then the groups are isomorphic if and only if
the monoids $\mathcal{B}(G)$ and $\mathcal{B}(G')$ are isomorphic.

The Isomorphism Problem II

Theorem

Let G and G' be abelian groups, and suppose that G is a direct sum of cyclic groups.
Then the groups are isomorphic if and only if the monoids $\mathcal{B}_{\pm}(G)$ and $\mathcal{B}_{\pm}(G')$ are isomorphic.

- There are isomorphisms between the monoids which do not stem from group isomorphisms.
- No groups are known for which the conclusion does not hold.

Outline

Unions of Sets of Lengths and Sets of Distances I

If $L = \{m_0, \ldots, m_k\} \subset \mathbb{Z}$ with $m_0 < \ldots < m_k$, then

$$\Delta(L) = \{m_i - m_{i-1} : i \in [1, k]\} \subset \mathbb{N}$$

is the set of distances of L, and

$$\Delta(H) = \bigcup_{a \in H} \Delta\big(\mathsf{L}(a)\big) \ \subset \ \mathbb{N}$$

the set of distances of H. For $k \in \mathbb{N}$, we call

$$\begin{aligned} \mathcal{U}_k(H) &= \bigcup_{k \in \mathsf{L}(a)} \mathsf{L}(a) \\ &= \{\ell \in \mathbb{N} \mid \text{there is an equation } u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_\ell\} \end{aligned}$$

the union of length sets containing k.

Unions .. and Sets of Distances II

Theorem

Let G be a finite abelian group.
The monoids $\mathcal{B}(G)$ and $\mathcal{B}_{\pm}(G)$ have the foll. property.

1. *The set of distances is an interval with minimum* 1 *and with* $\max \Delta\big(\mathcal{B}(G)\big) \le \mathsf{D}(G) - 2$ *and* $\max \Delta\big(\mathcal{B}_{\pm}(G)\big) \le \mathsf{D}_{\pm}(G) - 2$.
2. *For all $k \in \mathbb{N}$, the unions $\mathcal{U}_k(\cdot)$ are finite intervals.*

These results do not hold true for general

- Dedekind domains,
- orders in number fields,
- numerical monoids.

The Structure of Length Sets in finitely generated Monoids

Theorem (Freiman, G., Halter-Koch, Kainrath)

Let H be a finitely generated monoid.
There is a bound M and a finite set $\Delta^*(H)$ such that every length set $\mathsf{L}(a)$ is an AAMP with difference $d \in \Delta^*(H)$ and bound M, where

$$\Delta^*(H) = \{\min \Delta(S) \colon S \text{ is a divisor-closed submonoid of } H\}\,.$$

- If $H = \mathcal{B}(G)$, then $S = \mathcal{B}(G_0)$ for some $G_0 \subset G$.
- If $H = \mathcal{B}_{\pm}(G)$, then $S = \mathcal{B}_{\pm}(G_0)$ for some $G_0 \subset G$.

W.Schmid 2009 This description is best possible.

Infinite Abelian Groups

Theorem (G.+Kainrath 2024)

Let G be an infinite abelian group.
For every finite nonempty subset $L^* \subset N_{\geq 2}$,
there is a zero-sum sequence $S \in \mathcal{B}(G)$ such that

$$\mathsf{L}_{\mathcal{B}(G)}(S) = \mathsf{L}_{\mathcal{B}_{\pm}(G)}(S) = L^* \,.$$

Similar realization results hold true for rings of integer-valued polynomials, such as

$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \colon f(\mathbb{Z}) \subset \mathbb{Z}\} \subset \mathbb{Q}[X] \,.$$

Characterizations of Class Groups

Classic Philosophy in Algebraic Number Theory

The class group determines the arithmetic.

This was turned into results by the machinery of transfer hom's.

Narkiewicz 1974: Inverse problem

Does the arithmetic determine the class group?

- First affirmative answers were given in the 1980s.
- BUT: Which arithmetical properties should be used in the characterization?
- The best investigated properties are length sets.
- Are length sets sufficient to do the characterization ?

The Characterization Problem

Recall: $\mathcal{L}(H) = \{\mathsf{L}(a) \colon a \in H\}$.

The Characterization Problem for $\mathcal{B}(G)$.

Given two finite abelian groups G and G' such that $\mathcal{L}\big(\mathcal{B}(G)\big) = \mathcal{L}\big(\mathcal{B}(G')\big)$. Does it follow that $G \cong G'$?

The Characterization Problem for $\mathcal{B}_{\pm}(G)$.

Given two finite abelian groups G and G' such that $\mathcal{L}\big(\mathcal{B}_{\pm}(G)\big) = \mathcal{L}\big(\mathcal{B}_{\pm}(G')\big)$. Does it follow that $G \cong G'$?

A necessary condition for an affirmative answer holds true.

- $\mathcal{B}(G) \cong \mathcal{B}(G') \quad \Longleftrightarrow \quad G \cong G'$.
- $\mathcal{B}_{\pm}(G) \cong \mathcal{B}_{\pm}(G') \quad \Longleftrightarrow \quad G \cong G'$.

Comparing Systems of Length Sets

Let G be a finite abelian group.

- There are (up to isomorphism) only finitely many finite abelian groups G' such that

$$\mathcal{L}\big(\mathcal{B}(G)\big) = \mathcal{L}\big(\mathcal{B}(G')\big) .$$

- There are (up to isomorphism) only finitely many finite abelian groups G' such that

$$\mathcal{L}\big(\mathcal{B}_{\pm}(G)\big) = \mathcal{L}\big(\mathcal{B}_{\pm}(G')\big) .$$

On the Characterization Problem for $\mathcal{B}(G)$

Gao+G.+Schmid+Zhong

Suppose that $\mathcal{L}(\mathcal{B}(G)) = \mathcal{L}(\mathcal{B}(G'))$.
Then G *and* G' *are isomorphic in each of the following cases*:

1. $G = C_{n_1} \oplus C_{n_2}$, *where* $n_1, n_2 \in \mathbb{N}$ *with* $n_1 \mid n_2$ *and* $n_1 + n_2 > 4$.
2. $G = C_n^r$, *where* $r, n \in \mathbb{N}$ *are as follows* :
   - $r \le n - 3$
   - $r \ge n - 1$ *and* n *is a prime power.*
   - *ongoing work .....*

Crucial ingredients.

- We have $\mathsf{D}(C_{n_1} \oplus C_{n_2}) = n_1 + n_2 - 1$ and the structure of the minimal zero-sum sequences of maximal length is known.
- Information on $\Delta^*(\mathcal{B}(G))$.

On the Characterization Problem for $\mathcal{B}_{\pm}(G)$

Theorem (Fabsits+G.+Reinhart+Zhong)

Let G be cyclic of odd order.

If $\mathcal{L}\big(\mathcal{B}_{\pm}(G)\big) = \mathcal{L}\big(\mathcal{B}_{\pm}(G')\big)$, then G and G' are isomorphic.

Further new results by W. Schmid et al.: see the References.

Outline

References

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