Projective monomial curves and their affine projections

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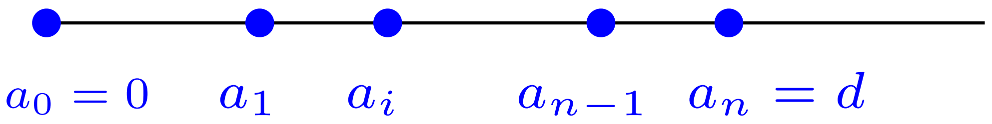
PROJECTIVE COHEN-MACAULAY MONOMIAL CURVES AND THEIR AFFINE CHARTS

IGNACIO GARCÍA-MARCO, PHILIPPE GIMENEZ, AND MARIO GONZÁLEZ-SÁNCHEZ

Abstract. In this paper, we explore when the Betti numbers of the coordinate rings of a projective monomial curve and one of its affine charts are identical. Given an infinite field k and a sequence of relatively prime integers $a_0 = 0 < a_1 < \cdots < a_n = d$, we consider the projective monomial curve $\mathcal{C} \subset \mathbb{P}_k^n$ of degree d parametrically defined by $x_i = u^{a_i} v^{d-a_i}$ for all $i \in \{0, \ldots, n\}$ and its coordinate ring $k[\mathcal{C}]$. The curve $\mathcal{C}_1 \subset \mathbb{A}_k^n$ with parametric equations $x_i = t^{a_i}$ for $i \in \{1, \ldots, n\}$ is an affine chart of $\mathcal{C}$ and we denote by $k[\mathcal{C}_1]$ its coordinate ring. The main contribution of this paper is the introduction of a novel (Gröbner-free) combinatorial criterion that provides a sufficient condition for the equality of the Betti numbers of $k[\mathcal{C}]$ and $k[\mathcal{C}_1]$. Leveraging this criterion, we identify infinite families of projective curves satisfying this property. Also, we use our results to study the so-called shifted family of monomial curves, i.e.,

Framework

$a_0 = 0 < a_1 < \cdots < a_{n-1} < a_n = d$ a sequence of relatively prime integers



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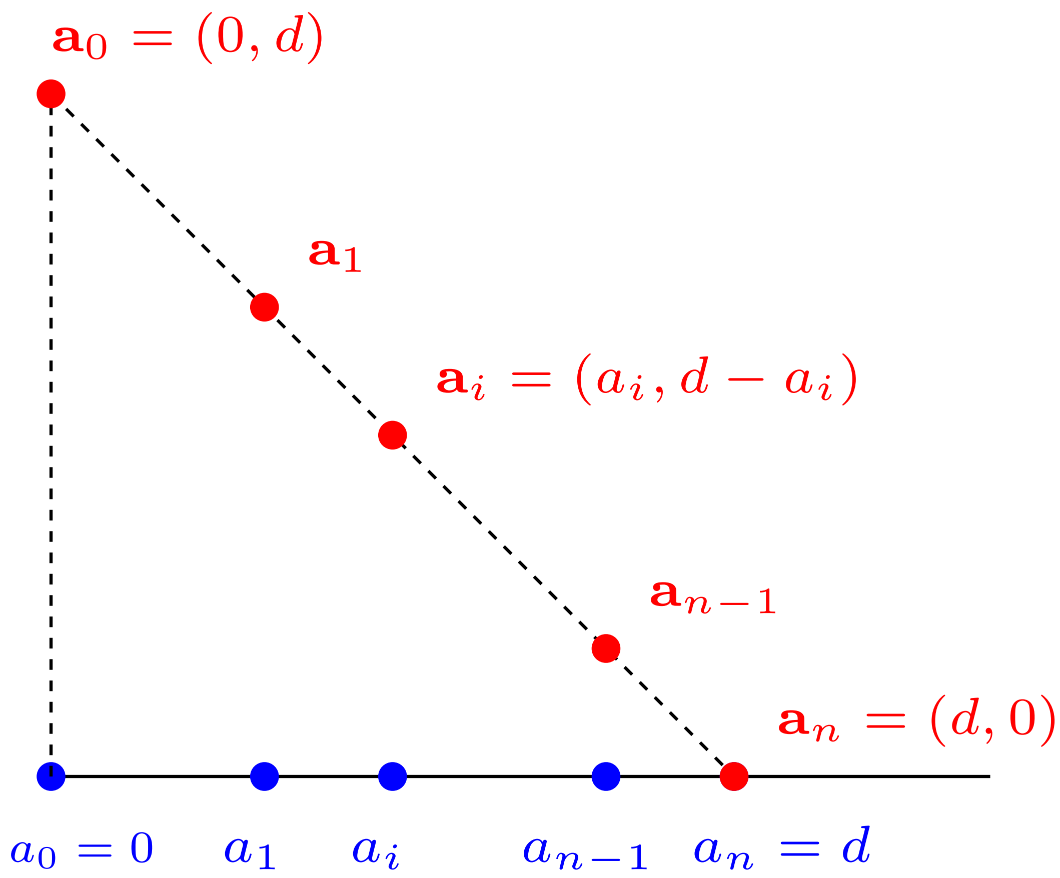

For $i = 0, \ldots, n-1$,
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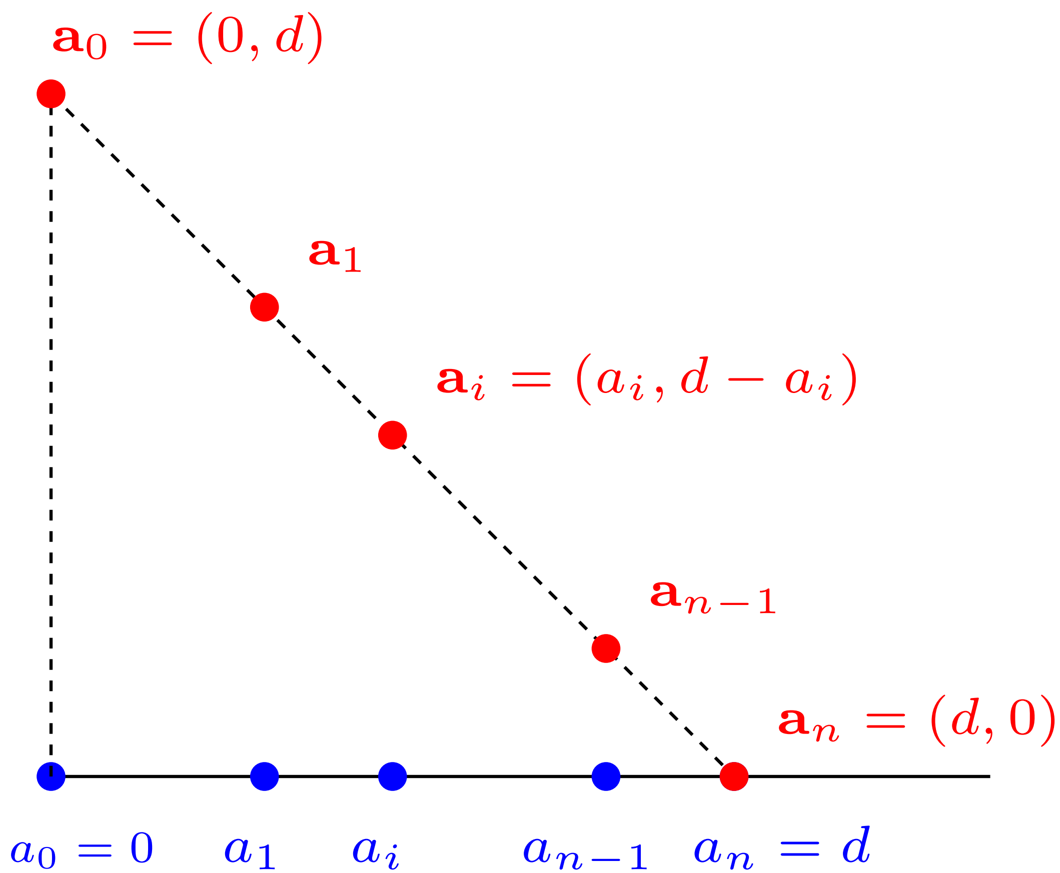

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$\mathcal{A}_1 = \{a_1, \ldots, a_n\} \subset \mathbb{N}$

Betti numbers (I)

- In general, given $\mathcal{B} = \{b_1, \ldots, b_n\} \subset \mathbb{N}^m$, a set of nonzero vectors, consider the monoid (semigroup) spanned by $\mathcal{B}$

$$\mathcal{S}_{\mathcal{B}} := \langle b_1, \ldots, b_n \rangle = \{\alpha_1 b_1 + \cdots + \alpha_n b_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{N}\} \subset \mathbb{N}^n$$

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- The toric ideal determined by $\mathcal{B}$: $I_\mathcal{B} = \ker \varphi_\mathcal{B}$
$\varphi_\mathcal{B} : k[\mathbf{x}] \longrightarrow k[\mathbf{t}]$ induced by $x_i \mapsto \mathbf{t}^{b_i}$.

$$k[\mathcal{S}_\mathcal{B}] \simeq k[\mathbf{x}]/I_\mathcal{B}$$

- $I_\mathcal{B}$ is a $\mathcal{S}_\mathcal{B}$-homogeneous binomial ideal

$$\deg_{\mathcal{S}_\mathcal{B}}(x_i) := b_i\,; \qquad \deg_{\mathcal{S}_\mathcal{B}}(\mathbf{x}^\alpha) := \alpha_1 b_1 + \cdots + \alpha_n b_n \in \mathcal{S}_\mathcal{B}$$

Betti numbers (II)

- One can consider a minimal $\mathcal{S}_{\mathcal{B}}$-graded free resolution of $k[\mathcal{S}_{\mathcal{B}}]$ as $\mathcal{S}_{\mathcal{B}}$-graded $k[\mathbf{x}]$-module

$$\mathcal{F} : 0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_0 \longrightarrow k[\mathcal{S}_{\mathcal{B}}] \longrightarrow 0$$

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- The i-th Betti number of $k[\mathcal{S}_{\mathcal{B}}]$ is $\beta_i(k[\mathcal{S}_{\mathcal{B}}]) = \mathrm{rank}(F_i)$; the Betti sequence of $k[\mathcal{S}_{\mathcal{B}}]$ is $(\beta_i(k[\mathcal{S}_{\mathcal{B}}])\,;\, 0 \leq i \leq p)$.

- $k[\mathcal{S}_{\mathcal{B}}]$ is Cohen-Macaulay when $\dim k[\mathcal{S}_{\mathcal{B}}] = \operatorname{depth} k[\mathcal{S}_{\mathcal{B}}]$.

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$k[\mathcal{S}]$ is the coordinate ring of a projective monomial curve, $\mathcal{C}$

$k[\mathcal{S}_1]$ is the coordinate ring of an affine monomial curve, $\mathcal{C}_1$ (an affine chart of $\mathcal{C}$)

An example: $0 < 5 < 6 < 7 < 8 < 9 < 10$

The numerical semigroup $\mathcal{S}_1$: $\mathcal{A}_1 = \{5, 6, 7, 8, 9, 10\}$

$$I_{\mathcal{A}_1} = \langle x_1 - u^5, x_2 - u^6, x_3 - u^7, x_4 - u^8, x_5 - u^9, x_6 - u^{10}\rangle \cap k[\mathbf{x}]$$
$$= \langle x_5^2 - x_4x_6,\ x_4x_5 - x_3x_6,\ \ldots,\ x_1^2 - x_6\rangle$$

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The affine semigroup $\mathcal{S}$:
$\mathcal{A} = \{(0, 10), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1), (10, 0)\}$

$$I_{\mathcal{A}} = \langle x_0 - v^{10}, x_1 - u^5 v^5, x_2 - u^6 v^4, x_3 - u^7 v^3, x_4 - u^8 v^2, x_5 - u^9 v$$
$$x_6 - u^{10} \rangle \cap k[x_0, \ldots, x_6]$$
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Betti sequence of

$0 < 1 < 3 < 4$

Betti seq. of $k[\mathcal{S}_1]$: $(1, 2, 1)$

Betti seq. of $k[\mathcal{S}]$: $(1, 4, 4, 1)$

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$\beta_i(k[\mathcal{S}]) \geq \beta_i(k[\mathcal{S}_1])$ for all i

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J. Saha, I. Sengupta and P. Srivastava. *Betti sequence of the projective closure of affine monomial curves*. J. Symb. Comput. **119** (2023)

[S^3] Theorem. Let $\mathcal{G}$ be the reduced Gröbner basis of $I_{\mathcal{A}_1}$ with respect to the degree reverse lexicographic (*degrevlex*) order with $x_1 > x_2 > \cdots > x_n$.
If $k[\mathcal{S}]$ is Cohen-Macaulay and x_n belongs to the support of all non-homogeneous binomials of $\mathcal{G}$, then $\beta_i(k[\mathcal{S}]) = \beta_i(k[\mathcal{S}_1])$, $\forall i$.

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[S^3] Condition

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In our previous example:

$$\mathcal{G} = \{x_5^2 - x_4x_6,\ x_4x_5 - x_3x_6,\ x_3x_5 - x_2x_6,\ x_2x_5 - x_1x_6$$

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We are looking for a **combinatorial** condition

Our tool: Apery Sets

Apery Set Ap_1

- $\mathcal{S}_1 = \langle a_1, \ldots, a_n \rangle$
- $\mathrm{Ap}_1 := \{y \in \mathcal{S}_1 \mid y - d \notin \mathcal{S}_1\}$
- $(\mathrm{Ap}_1, \leq_1)$ is a poset, where $y \leq_1 z \Leftrightarrow z - y \in \mathcal{S}_1$.

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  $|\mathrm{AP}_{\mathcal{S}}| \geq d$ and $k[\mathcal{S}]$ is Cohen-Macaulay $\Leftrightarrow |\mathrm{AP}_{\mathcal{S}}| = d$
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Let $(P, \leq)$ be a finite poset

For $y, z \in P$, $y \prec z \Leftrightarrow y < z$ and there is no w s.t. $y < w < z$

P is graded if there is a function $\rho : P \to \mathbb{N}$ s.t. $\rho(z) = \rho(y) + 1$ if $y \prec z$

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  $\mathrm{AP}_{\mathcal{S}}$ is always graded

Main Result

Theorem [G^3]
$(\mathrm{Ap}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1) \Rightarrow \beta_i(k[\mathcal{S}]) = \beta_i(k[\mathcal{S}_1])$ for all i.

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Idea of the proof:

Proposition [G^3]. The following are equivalent:

- The posets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{Ap}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are isomorphic;
- $|\mathrm{AP}_{\mathcal{S}}| = d$, $(\mathrm{Ap}_1, \leq_1)$ is graded & $\{a_1, \ldots, a_{n-1}\} \subset \mathrm{MSG}(\mathcal{S}_1)$.
- [S^3] condition.

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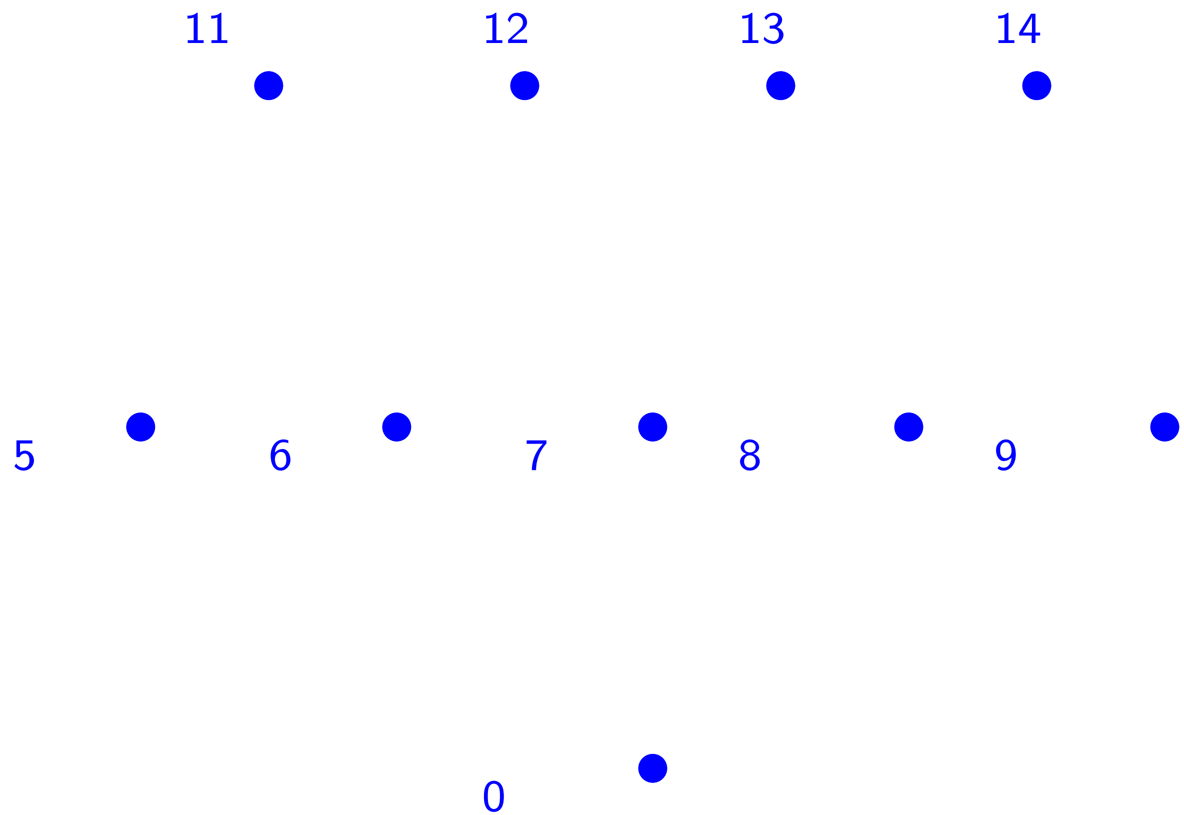
$(\mathrm{Ap}_1, \leq_1)$ is graded iff $\mathrm{Ap}_1 \subset \mathrm{ULF}(\mathcal{S}_1)$

Example: $0 < 5 < 6 < 7 < 8 < 9 < 10$

$\mathrm{Ap}_1 = \{0, 5, 6, 7, 8, 9, 11, 12, 13, 14\}$



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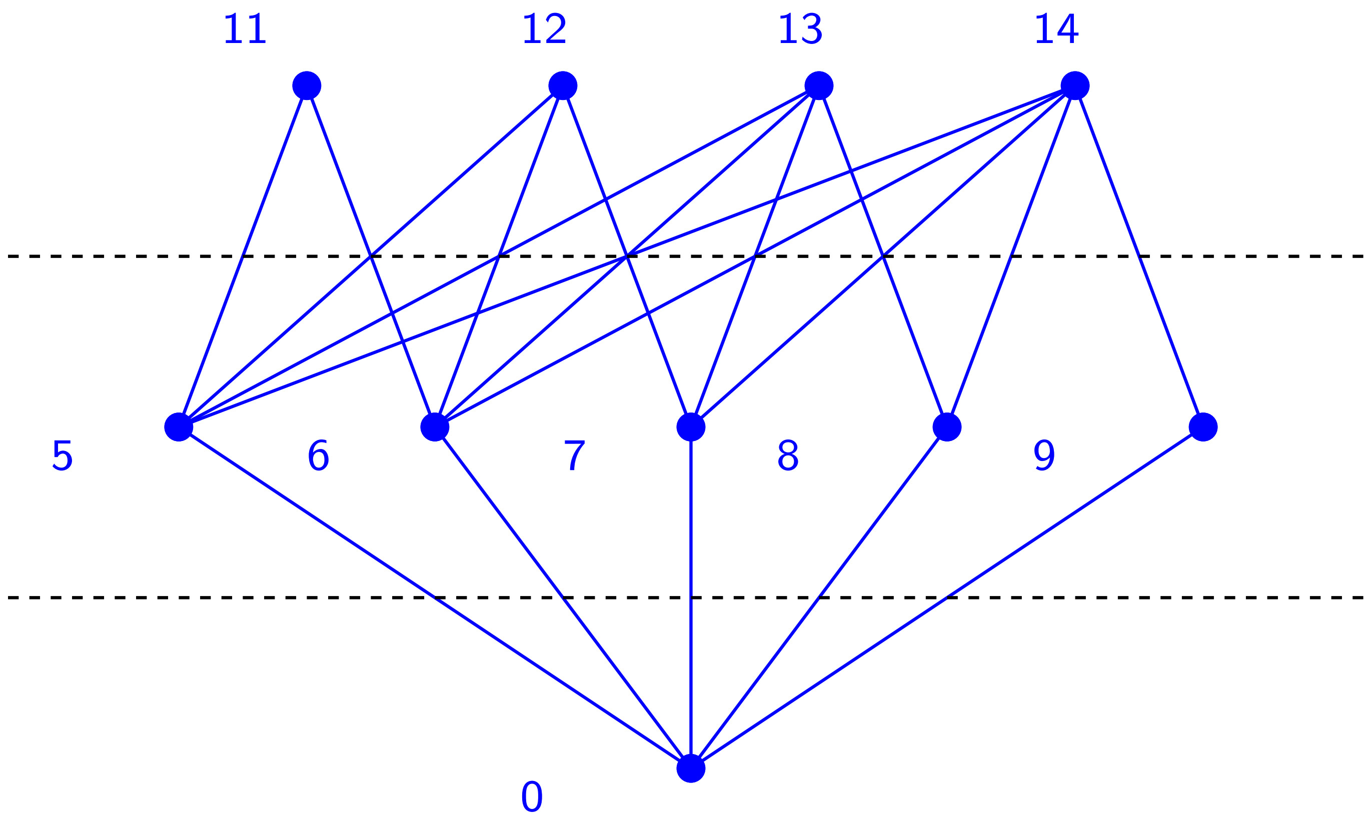

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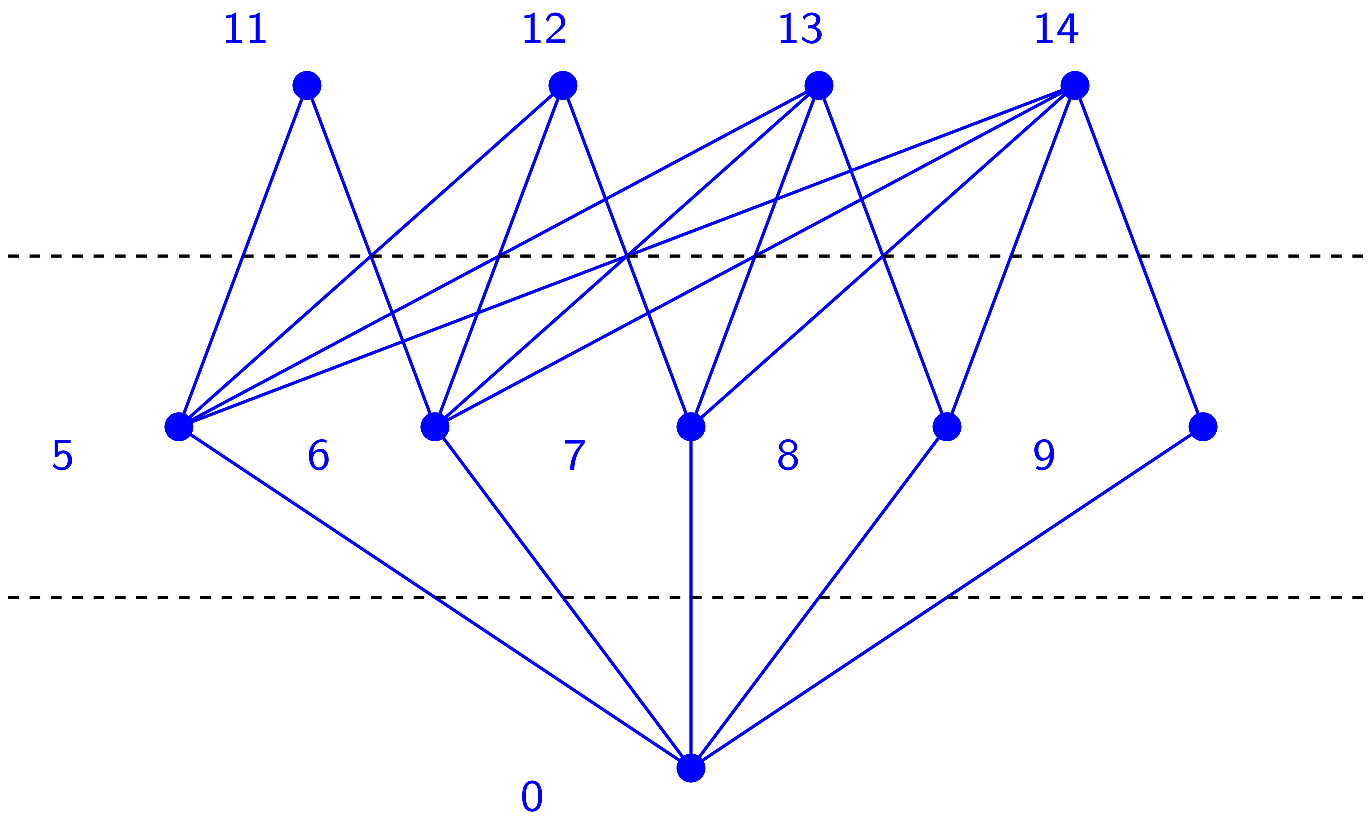

$\mathrm{AP}_{\mathcal{S}} = \{(0,0), (5,5), (6,4), (7,3), (8,2), (9,1), (11,9),$
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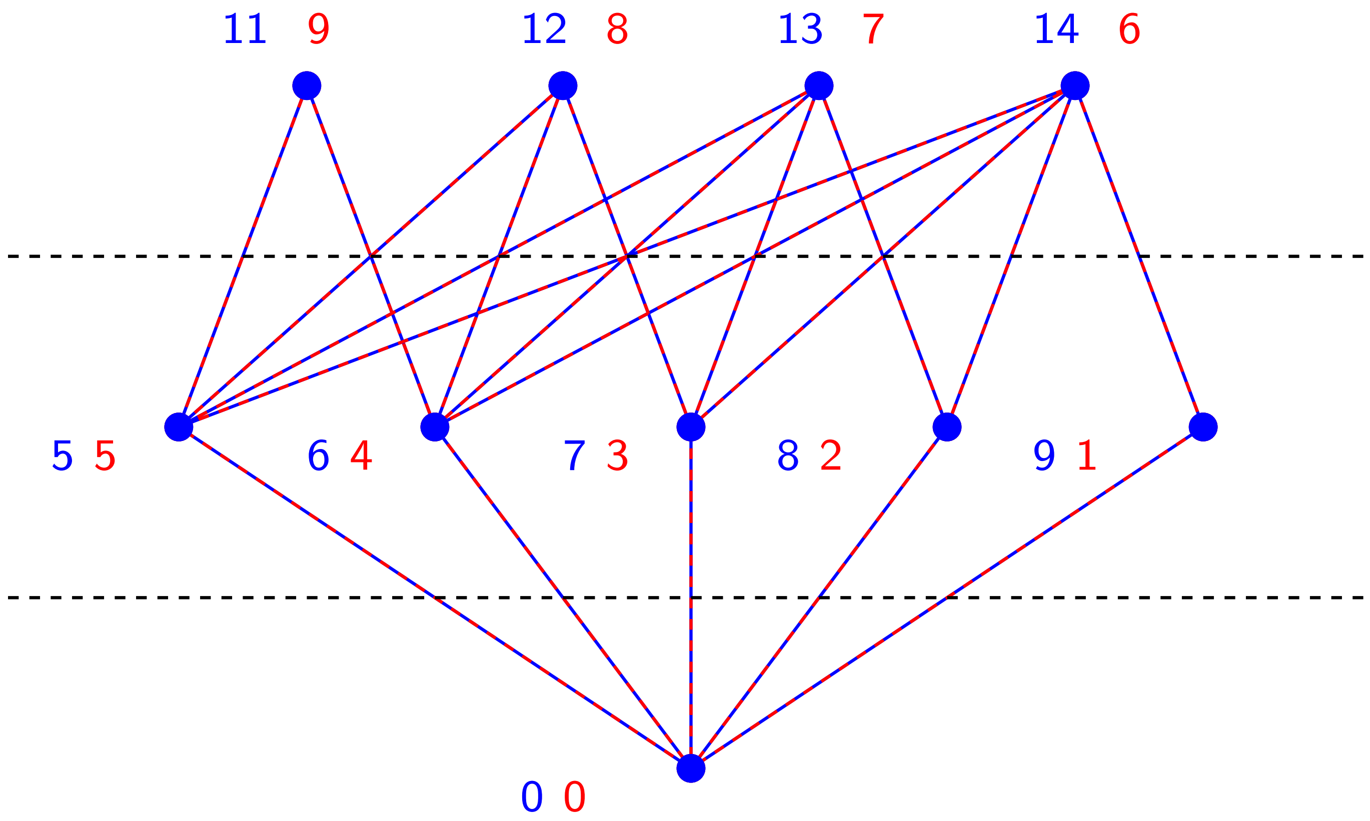

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Arithmetic sequence

$$0 < a_1 < a_1 + e < a_1 + 2e < \cdots < a_1 + (n-1)e\ , \qquad \gcd(a_1, e) = 1$$

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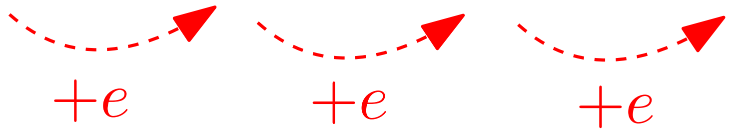

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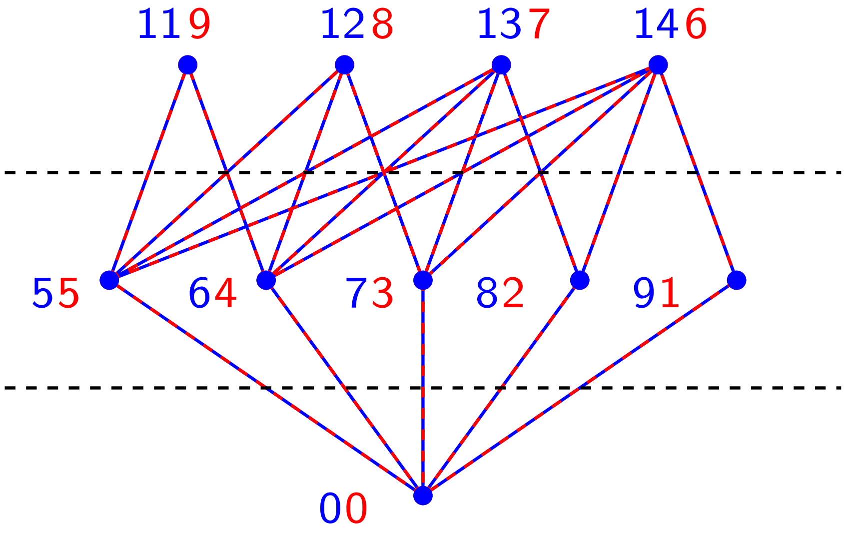

Canonical projections of arithmetic sequences

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Consider $\mathcal{A}_1 = \{a_1, \ldots, a_n\} \setminus \{a_r\}$ and
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Proposition [G^3]

$$\mathrm{Ap}_{\mathcal{S}} \simeq \mathrm{Ap}_1 \iff \begin{cases} a_1 > n-2 \text{ and } a_1 \neq n, & \text{if } r = 2, \\ a_1 \geq n \text{ and } r \leq a_1 - n + 1, & \text{if } 3 \leq r \leq n-2, \\ a_1 \geq n-2, & \text{if } r = n-1. \end{cases}$$

Hence, if the previous condition holds, then $\beta_i(k[\mathcal{S}_1]) = \beta_i(k[\mathcal{S}])$, $\forall i$.

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1	✓	$(1, 9, 16, 9, 1)$	$(1, 9, 16, 9, 1)$
6	✓	$(1, 10, 20, 15, 4)$	$(1, 10, 20, 15, 4)$

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3	X	$(1, 7, 14, 11, 3)$	$(1, 7, 17, 16, 5)$
4	X	$(1, 6, 11, 8, 2)$	$(1, 7, 17, 16, 5)$
5	✓	$(1, 6, 10, 6, 1)$	$(1, 6, 10, 6, 1)$
6	✓	$(1, 10, 20, 15, 4)$	$(1, 10, 20, 15, 4)$

¡Gracias!

Thank you!