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The value semigroup of a plane curve singularity with several branches

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0. Algebroid branches and curves

Algebroid branch: one-dimensional domain of the form
$R = k[[x_1, \ldots, x_n]]/P$ (k algebraically closed).

$Q(R) \cong k((t))$ and $\overline{R} \cong k[[t]]$ (and it is a finite R-module)
and $v(R \setminus \{0\})$ is a numerical semigroup.

Algebroid curve: one-dimensional, reduced ring of the form
$R = k[[x_1, \ldots, x_n]]/P_1 \cap \cdots \cap P_h$
(P_i height $n-1$ primes, k algebraically closed).
$R_i = k[[x_1, \ldots, x_n]]/P_i$ is the i-th branch of R.

$Q(R) \cong k((t_1)) \times \cdots \times k((t_h))$ and $\overline{R} \cong k[[t_1]] \times \cdots \times k[[t_h]]$.
If we set $v(r) = (v_1(r_1), \ldots, v_h(r_h))$, then the value semigroup is:

$$S = v(R) := \{v(r) : \ r \in R, \ r \text{ non-zero divisor}\} \subset \mathbb{N}^h.$$

1. Value semigroups and equisingularity of plane curves.

Value semigroup is a possible criterion of equisingularity for algebroid branches or curves.

Two plane algebroid branches are formally equivalent (i.e. they have the same multiplicity sequence) $\Leftrightarrow$ they have the same value semigroup.
In case $k = \mathbb{C}$ two plane analytic branches are topologically equivalent $\Leftrightarrow$ are formally equivalent [Zariski].

Both multiplicity sequences and value semigroups of plane algebroid branches have been characterized [Zariski, Bertin-Carbonne, Brezinsky, Angermüller].

As in the one branch case, two plane algebroid curves are formally equivalent $\Leftrightarrow$ they have the same value semigroup [Waldi].

Garcia (2 branches case) and Delgado gave a characterization of value semigroups of plane curves depending on its projections.

We want to give a constructive characterization connected to the blowing up process, based on an old result of Apéry (that holds for the 1 branch case), directly relating the value semigroup and the "multiplicity tree".

2 Value semigroups of algebroid curves

The value semigroup of an algebroid curve is a submonoid of $\mathbb{N}^h$, with some more properties connected to valuations.

In the case $h = 2$, setting
$\Delta^S(a_1, a_2) = (\{(a_1, y) : a_2 < y\} \cup \{(x, a_2) : a_1 < x\}) \cap S$, they are:

(1) $\exists\ \boldsymbol{\gamma} = \boldsymbol{\gamma}(S) \in \mathbb{N}^2$ s.t. $\boldsymbol{\Delta}^S(\boldsymbol{\gamma}) = \emptyset$ and $\boldsymbol{\gamma} + (1,1) + \mathbb{N}^2 \subseteq S$;

(2) $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S \ \Rightarrow \min(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S$;

(3)

(4) $(0, 0)$ is the only element of S on the axes.

$$R = \frac{k[[x,y,z]]}{(x^3 - z^2, y) \cap (x^3 - y^4, z)}$$

$$x \mapsto (t^2, u^4)$$

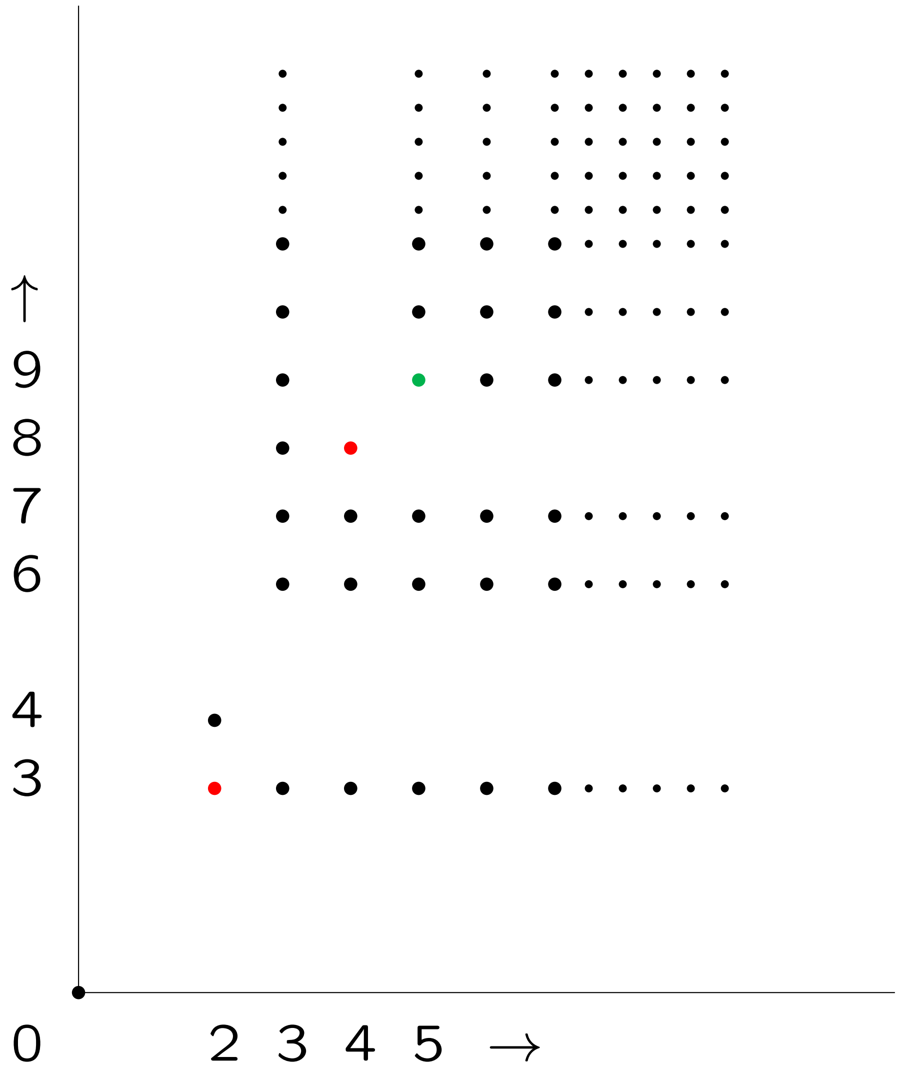
$$y \mapsto (0, u^3)$$

$$z \mapsto (t^3, 0)$$

$$v(x + y) = (2, 3)$$

$$\gamma = (4, 8)$$

$$\gamma + (1, 1) = (5, 9)$$



Picture 1. $S = v(R)$

3. Good semigroups

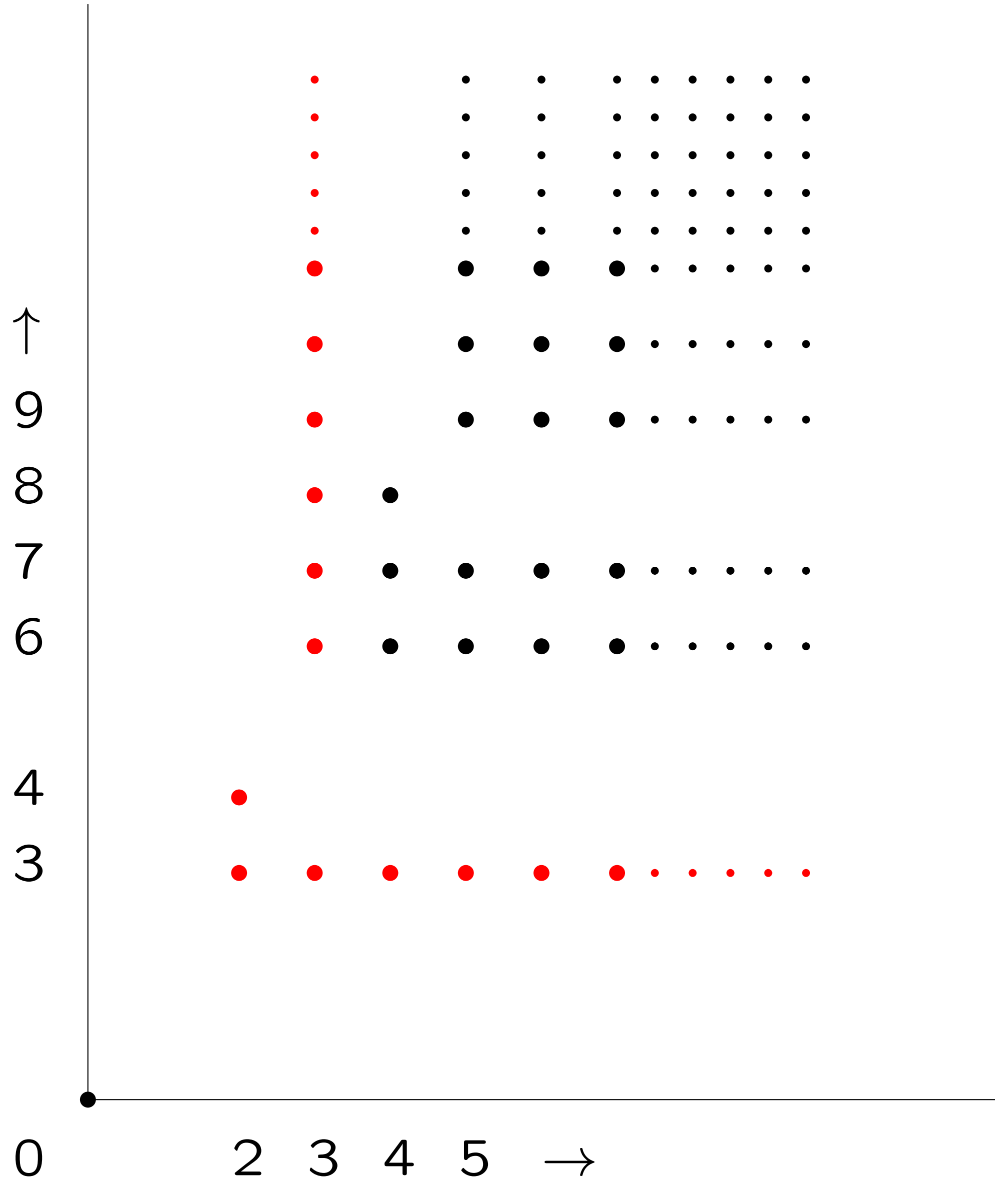
A subsemigroup S of $\mathbb{N}^h$ satisfying properties (1), (2), (3) is called a good semigroup. If (4) holds, it is said to be local.
Not all good semigroups arise as value semigroups [V. Barucci, _ , R. Fröberg - 2000], [N. Maugeri, G. Zito - 2019]
Open pb.: characterize value semigr. among good semigroups.

Definition. Relative ideal: $E \subseteq \mathbb{Z}^h$ s.t. $\boldsymbol{\alpha} + E \subseteq E$, $\forall\, \boldsymbol{\alpha} \in S$ and $\exists\, \boldsymbol{\alpha} \in S$, s.t. $\boldsymbol{\alpha} + I \subseteq S$. I is good if it satisfies (2), (3)
Remark: I fractional ideal of $R \Rightarrow v(I)$ good rel. ideal of $v(R)$.

"Bad" facts:

- good semigroups are not finitely generated as semigroups;
- good ideals are not finitely generated as semigroup ideals;

- operations on good ideals do not produce good ideals;
- It is much more difficult to prove results for $h \geq 3$, than for $h = 2$.



Picture 2. Generators of S

4. Why to study value semigroups and good semigroups?

It is possible to define a "distance" function,
$d(E \setminus F)$, between good relative ideals $E \supseteq F$ such that

Proposition. [_] If $I \supseteq J$ are two fractional ideals of R, then $\lambda_R(I/J) = d(v(I) \setminus v(J))$.
Then we can study the properties of R, using $v(R)$; e.g.:

• If S is local, set $\boldsymbol{e} = (e_1, \dots, e_h) = \min S \setminus \{\mathbf{0}\}$;
multiplicity: $\lambda_R(R/(x)) = e_1 + \cdots + e_h$
with x minimal reduction of $\mathfrak{m}$ i.e. $v(x) = \boldsymbol{e}$.
Notice that e_i is the multiplicity of the i-th branch of R;

• degree of singularity: $\lambda_R(\overline{R}/R) = d(\mathbb{N}^h \setminus S)$.

Also we can get information e.g on Goresteinness [Campillo, Delgado, Kiyek], Arf property, embedding dimension [Maugeri, Zito], type [_, Guerrieri, Micale].

5. Blowing up tree and multiplicity tree

Let R be a branch: its blow-up (or strict quadratic transform) is

$$R^{\mathfrak{m}} = \cup_{n>0}(\mathfrak{m}^n :_{Q(R)} \mathfrak{m}^n) = \mathfrak{m}^{n_0} :_{Q(R)} \mathfrak{m}^{n_0} \quad (\exists\ n_0 >> 0).$$

If x is a minimal reduction of $\mathfrak{m}$ and $\mathfrak{m} = (x, x_2, \ldots, x_\nu)$,
$R^{\mathfrak{m}} = R[x_2/x, \ldots x_\nu/x]$.

$R \subset R^{\mathfrak{m}} \subseteq \overline{R} \cong k[[t]]$, hence, denoting $R^{\mathfrak{m}} = R_1$, we can blow up its maximal ideal and so on, getting:

$$R = R_0 \subset R_1 \subset \cdots \subset R_l = \overline{R} = \overline{R} = \cdots$$

The sequence of multiplicities $e_i = e(R_i)$ is the multiplicity sequence of R.

More generally, if R is a curve and I an ideal of R, the blowing up R^I of I is $\cup_{n>0}(I^n :_{Q(R)} I^n) = I^{n_0} :_{Q(R)} I^{n_0}$ for some n_0.

Again we can associate to R a sequence (Lipman sequence) of semilocal rings

$$R = R_0 \subset R_1 \subset \cdots \subset R_l = \overline{R} = \overline{R} = \cdots$$

where R_{i+1} is obtained from R_i by blowing up the Jacobson radical of R_i, $\mathrm{J}(R_i)$.

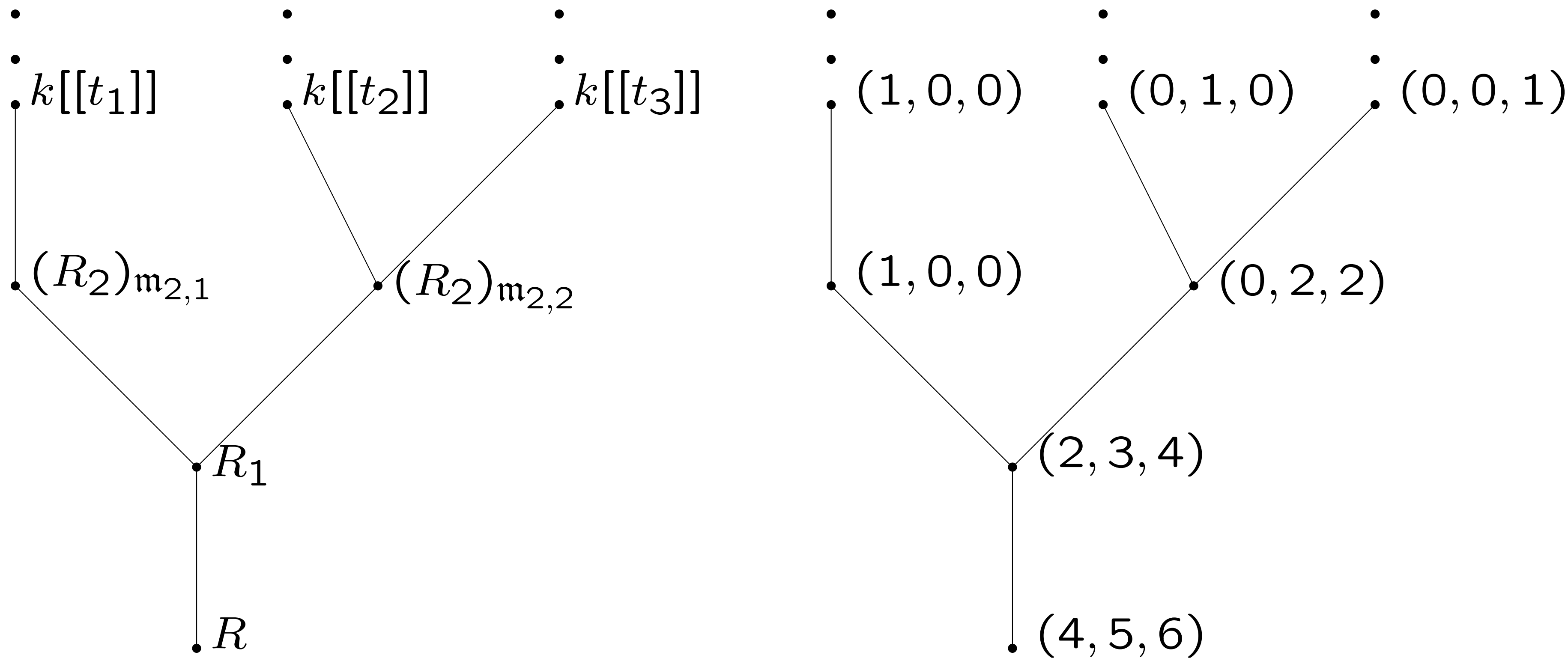
Fact. If $\mathfrak{m}_{i,1}, \ldots, \mathfrak{m}_{i,r_i}$ are the maximal ideals of R_i, then

$$R_i \cong R_{\mathfrak{m}_{i,1}} \times \cdots \times R_{\mathfrak{m}_{i,r_i}}$$

Hence to an algebroid curve R with $\overline{R} = V_1 \times \cdots \times V_h$ we can associate its blowing up tree of R and its multiplicity tree

Example.

$R = k + (t^4, u^5, v^6)k + (t^6k[[t]] \times ((u^8, v^{10})k + (u^{10}k[[u]] \times v^{12}k[[v]])))$



Picture 3

For non-plane singularities it is NOT possible to reconstruct the multplicity tree only by the value semigroup, nor viceversa.

6. Apéry set and value semigroups of plane branches

Let $s \in S \subseteq \mathbb{N}$. The Apéry set of S (with respect to s) is:

$$Ap(S,s) = \{x \in S : x - s \notin S\} = \{a_0 = 0 < a_1 < \cdots < a_{s-1} = f(S) + s\}$$

Theorem. [Apéry] [Angermüller]
Let R be a plane algebroid branch, $e = e(R)$ and $v(R) = S$.
Set $Ap(S,e) = \{a_0 = 0 < a_1 < a_2 < \ldots < a_{e-1}\}$; then

$$Ap(v(R_1), e) = \{a_0 < a_1 - e < a_2 - 2e < \ldots < a_{e-1} - (e-1)e\}.$$

$\Rightarrow$ Computation of the multiplicity sequence of a plane branch, by its value semigroup and vice-versa.

Example. $R = k[[t^4, t^6 + t^7]]$ (char$(k) \neq 2$). Set $S_1 = v(R_1)$.

$v(R) = S = \langle 4, 6, 13 \rangle \quad e(R) = 4 \quad Ap(S, 4) = \{0, 6, 13, 19\}$.

$\Rightarrow \ Ap(S_1, 4) = \{0, 2 = 6 - 4, 5 = 13 - 8, 7 = 19 - 12\}$, $S_1 = \langle 2, 5 \rangle$

Repeating the procedure we get the multiplicity sequence of R: $4, 2, 2, 1, \ldots$.

If we start with the multiplicity sequence, we can go backwards in the sequence of blowups:

assume to know that $S_1 = v(R_1) = \langle 2, 5 \rangle$,

$e_0 = 4$: $Ap(S_1, 4) = \{0, 2, 5, 7\}$

$\Rightarrow \qquad Ap(S, 4) = \{0, 6 = 2 + 4, 13 = 5 + 8, 19 = 7 + 12\}$

$\Rightarrow S = \langle 4, 6, 13, 19 \rangle = \langle 4, 6, 13 \rangle$.

The reason is:

$R = k[[X,Y]]/(F) = k[[x,y]] = k[[x]] + k[[x]]y + \cdots + k[[x]]y^{e-1}$,

where $x = X + (F)$, $y = Y + (F)$, $v(y) > v(x) = e$.

and, if $Ap(S,e) = \{a_0 = 0 < a_1 < a_2 < \ldots < a_{e-1}\}$, then

$$a_i = v(y^i + \phi_i(x,y))$$

where $\deg_y(\phi_i) < i$.

In the above example:

$R = k[[t^4, t^6 + t^7]]$, $x = t^4$, $y = t^6 + t^7$, $Ap(S,4) = \{0, 6, 13, 19\}$;

$a_1 = 6 = v(y)$, $a_2 = 13 = v(y^2 - x^3)$, $a_3 = 19 = v(y^3 - x^3 y)$.

$R_1 = R[y/x] = k[[x, y/x]] = k[[x]] + k[[x]](y/x) + \cdots + k[[x]](y/x)^{e-1}$

In the above example:

$R_1 = k[[t^4, t^2 + t^3]]$, $Ap(v(R_1), 4) = \{0, 2, 5, 7\}$, and, e.g.

$5 = v((y^2 - x^3)/x^2))$.

Why can we go backwards?

Proposition. [Barucci, _, Fröberg] Let R be a branch.
Set $R_1 = R[y/x]$, $e = v(x)$ and $Ap(S_1, e) = \{a'_0, \dots, a'_{e-1}\}$. Then

$$\exists\ g \in R_1\ :\ R_1 = k[[x]] + k[[x]]g + \cdots + k[[x]]g^{e-1}$$

and $a'_i = v(g^i + \psi_i)$ (with $\deg(\psi_i) < i$).

Moreover, $R = k[[x]] + k[[x]]gx + \cdots + k[[x]]g^{e-1}x^{e-1}$ and $\{(g^i + \psi_i)x^i \mid i = 0, \dots e-1\}$ determines the Apéry set of $v(R)$.

7. Apéry set and value semigroups of plane curves

Let $S \subset \mathbb{N}^h$ and set $\boldsymbol{\delta} = (d_1, \ldots, d_h) \in S$.
The Apéry set of S (with respect to $\boldsymbol{\delta}$) is:

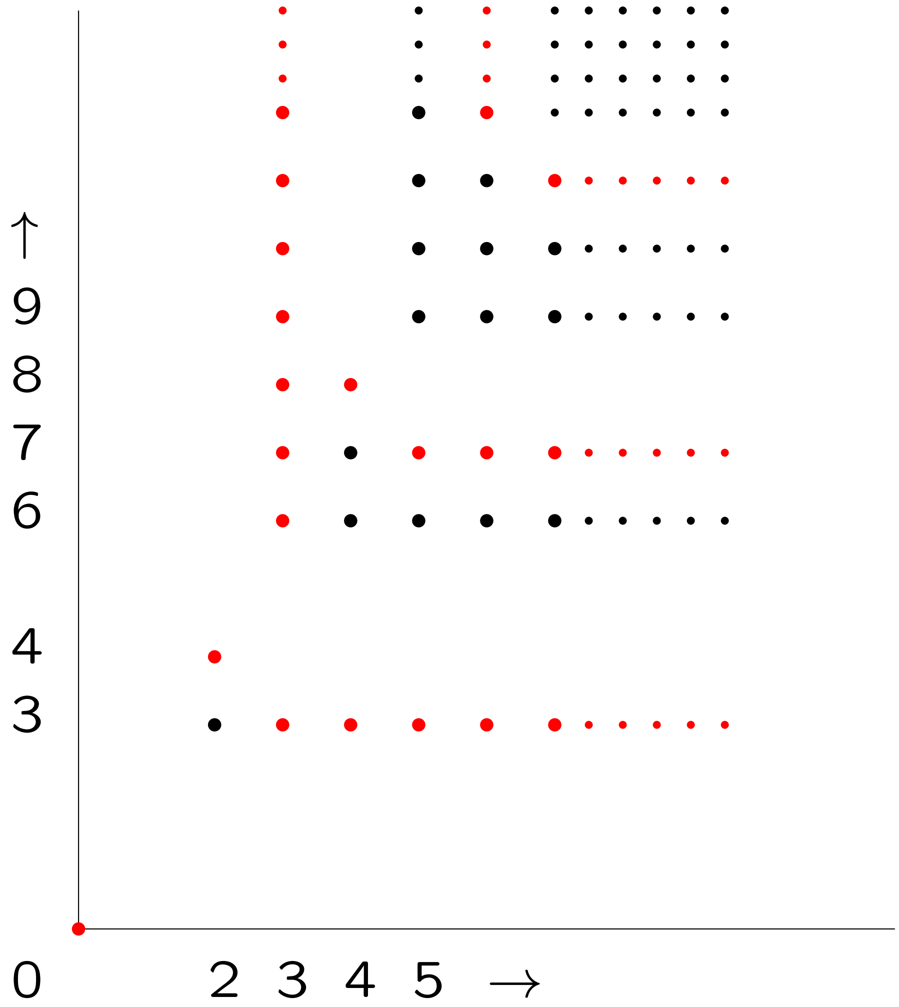
$$Ap(S, \boldsymbol{\delta}) = \{\boldsymbol{\alpha} \in S : \ \boldsymbol{\alpha} - \boldsymbol{\delta} \notin S\}$$

The problem, now, is that $Ap(S, \boldsymbol{\delta})$ is infinite and not linearly ordered.

We would like to have a partition of $Ap(S, \boldsymbol{\delta})$ in $D = d_1 + \cdots + d_h$ subsets:

$$Ap(S, \boldsymbol{\delta}) = \bigcup_{i=0}^{D-1} A_i$$

in such a way that the A_i play the role of the a_i.



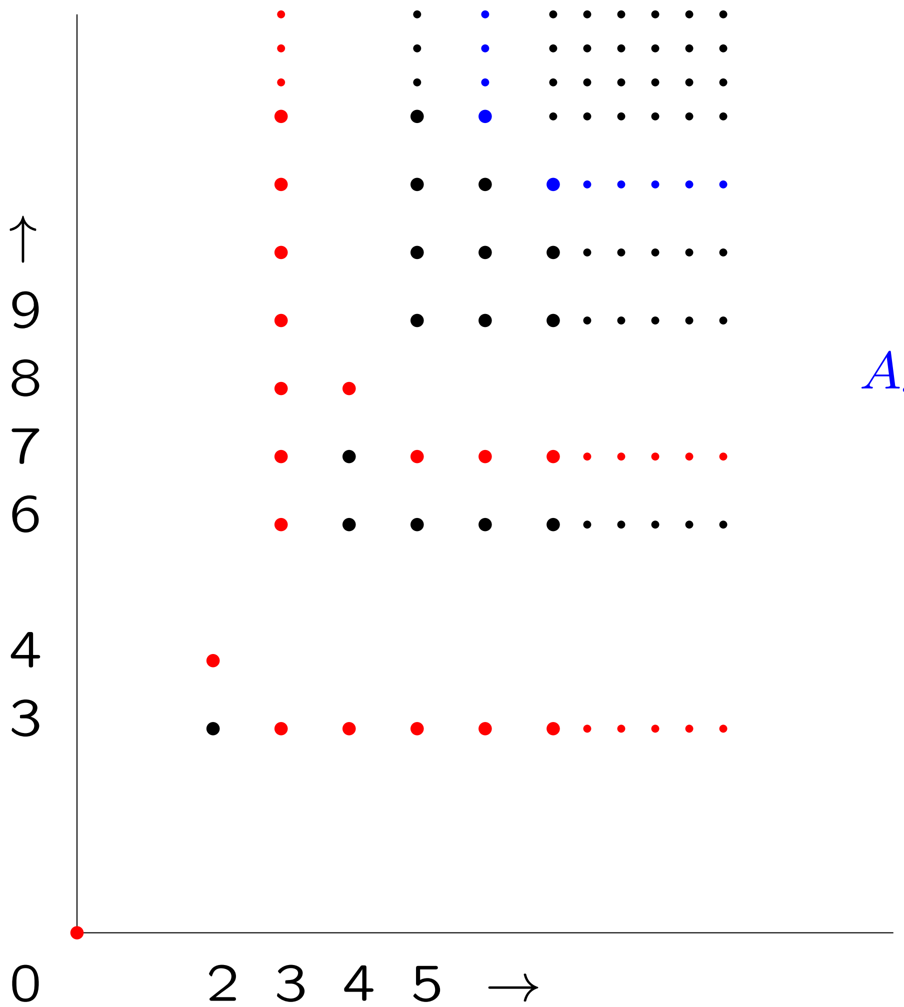
$\boldsymbol{\delta} = (2, 3) = \boldsymbol{e}$

now $Ap(S, \boldsymbol{\delta})$ is infinite

Picture 4. $Ap(S, \boldsymbol{\delta})$

How do we define the A_i?

Define $\boldsymbol{\alpha} \leq\leq \boldsymbol{\beta}$ iff either $\boldsymbol{\alpha} = \boldsymbol{\beta}$ or $\alpha_i < \beta_i$ for both $i = 1, 2$.

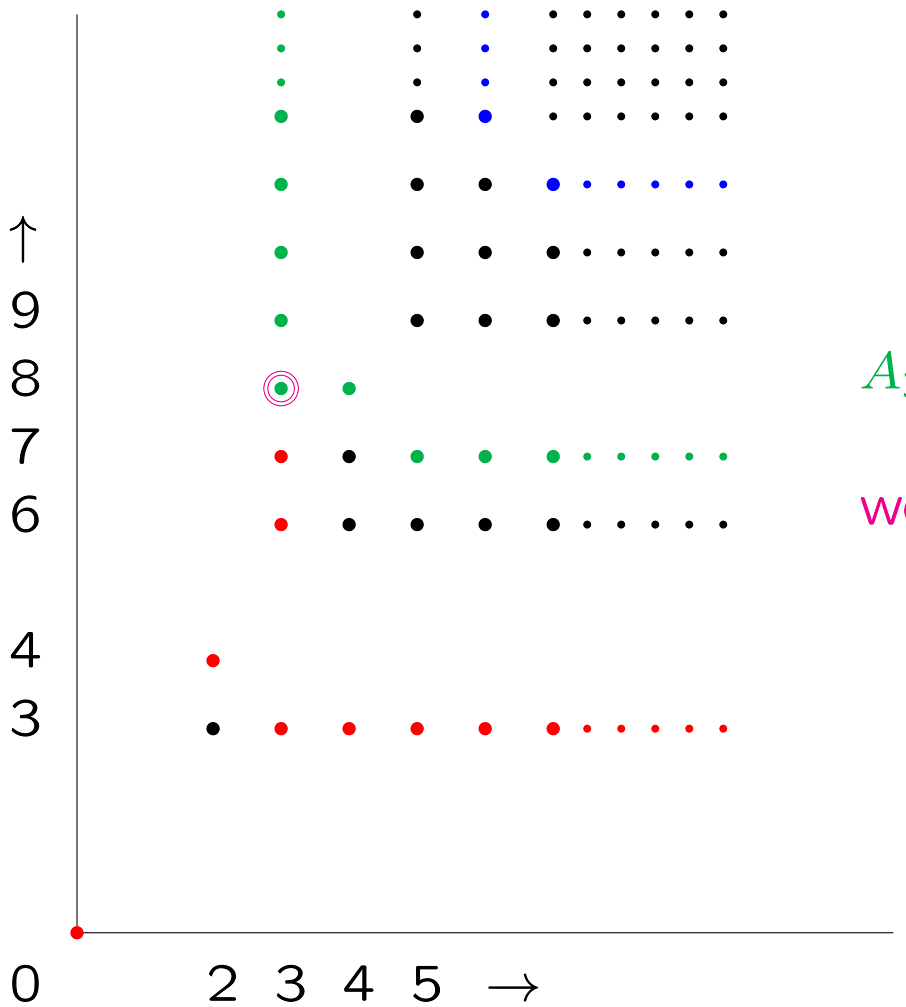


$$\boldsymbol{\delta} = \boldsymbol{e} = (2,3), \quad D = 2+3 = 5$$

$$A_4 = \{\boldsymbol{\alpha} \in Ap(S,\boldsymbol{e}) \mid \boldsymbol{\alpha} \text{ max. w.r.t. } \leq\leq\}$$

$$= \Delta^S(\gamma + \boldsymbol{e})$$

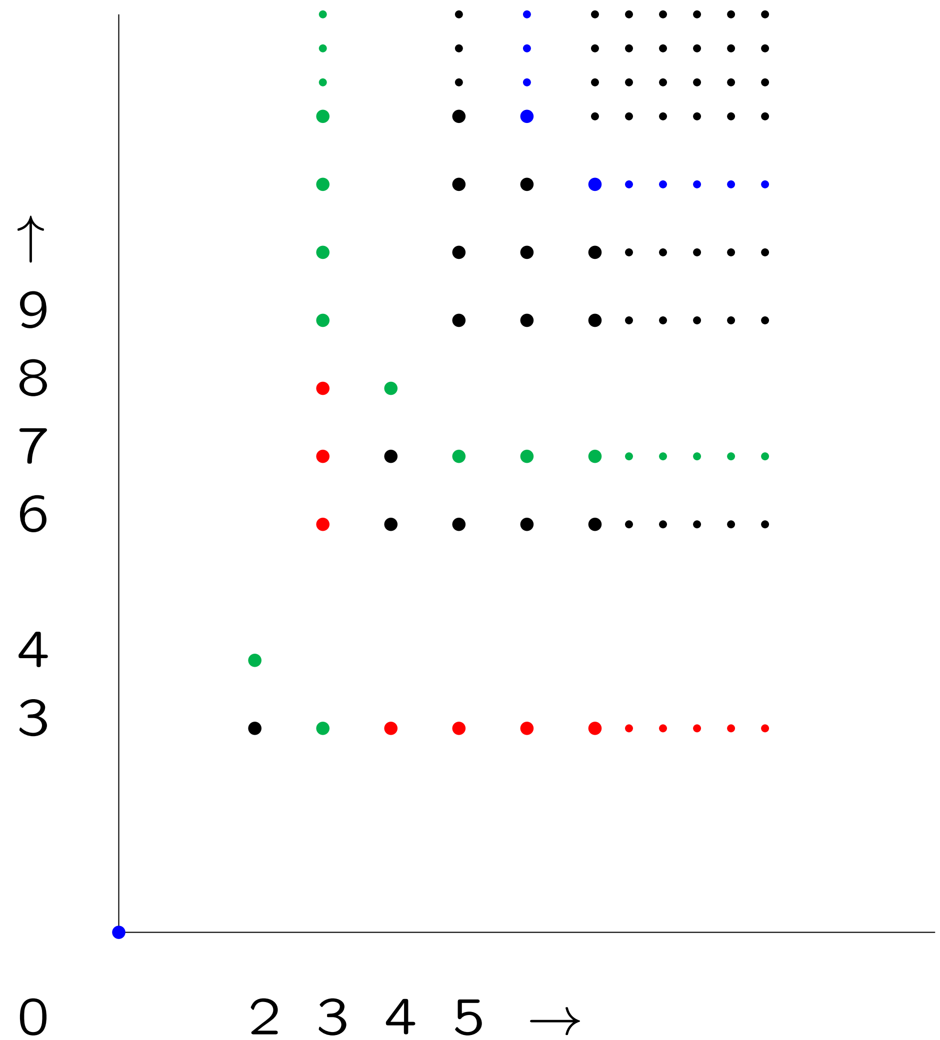
Picture 5. A_4



$A_3 \subseteq \{\boldsymbol{\alpha} \in Ap(S, \boldsymbol{e}) \setminus A_4$ max. w.r.t. $\leq\leq\}$

we exclude the $\boldsymbol{\beta}$ obtained as infimums

Picture 6. A_3



Picture 7. $Ap(S, \boldsymbol{e}) = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$

Theorem [_, Guerrieri, Micale] [Guerrieri, Maugeri, Micale]
Let $S \subseteq \mathbb{N}^h$ be a good semigroup, $\boldsymbol{\delta} = (d_1, \ldots, d_h)$ and set $D = d_1 + \cdots + d_h$. Then

$$Ap(S, \boldsymbol{\delta}) = \bigcup_{i=0}^{D-1} A_i.$$

8. Apéry process for plane curves

Let $R = k[[X, Y]]/(F)$; with $F = G_1 G_2 \cdots G_h$, (G_i irreducible, pairwise distinct). We can assume: $F = Y^E + \sum_{i=0}^{E-1} c_i(X)Y^i$, with $E = e(R)$.

Setting $x = X + (F)$ and $y = Y + (F)$:
$R = k[[x, y]] = k[[x]] + k[[x]]y + \cdots + k[[x]]y^{E-1}$,
where $v(y) > v(x) = \boldsymbol{e} = (e_1, \ldots, e_h)$ and $E = e_1 + \cdots + e_h$.

Theorem. [Barucci, _ , Fröberg] It is possible to define $T_i \subset R$, depending on y^i, such that, if $Ap(v(R), \boldsymbol{e}) = \cup_{i=0}^{E-1} A_i$, then

$$A_i = v(T_i)$$

So we can generalize the one branch case for R and R_1 both local.

If R_1 is not local how can we go backwards?

Proposition. [Guerrieri, Maugeri, Micale] It is possible to define the levels of the Apéry set in the non local case and describe them in function of the levels of the projections.

Theorem.[_, Delgado, Guerrieri, Maugeri, Micale] Let W non local, $\overline{W} = k[[t_1]] \times \cdots \times k[[t_h]]$, fix $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_h) \in v(W)$, with $\epsilon_i > 0$ $\forall i$; set $E = \epsilon_1 + \cdots + \epsilon_h$. Then

$\forall f = (f_1, f_2) \in W$, of value $v(f) = \boldsymbol{\epsilon}$, $\exists\ g = (g_1, g_2) \in W$, s.t.

$$W = k[[f]] + k[[f]]g + \cdots + k[[f]]g^{E-1}.$$

Theorem. [_, D, G, M, M] It is possible to define $T_i \subset W$, depending on g^i, such that, if $Ap(v(W), \boldsymbol{\epsilon}) = \cup_{i=0}^{E-1} A_i$,

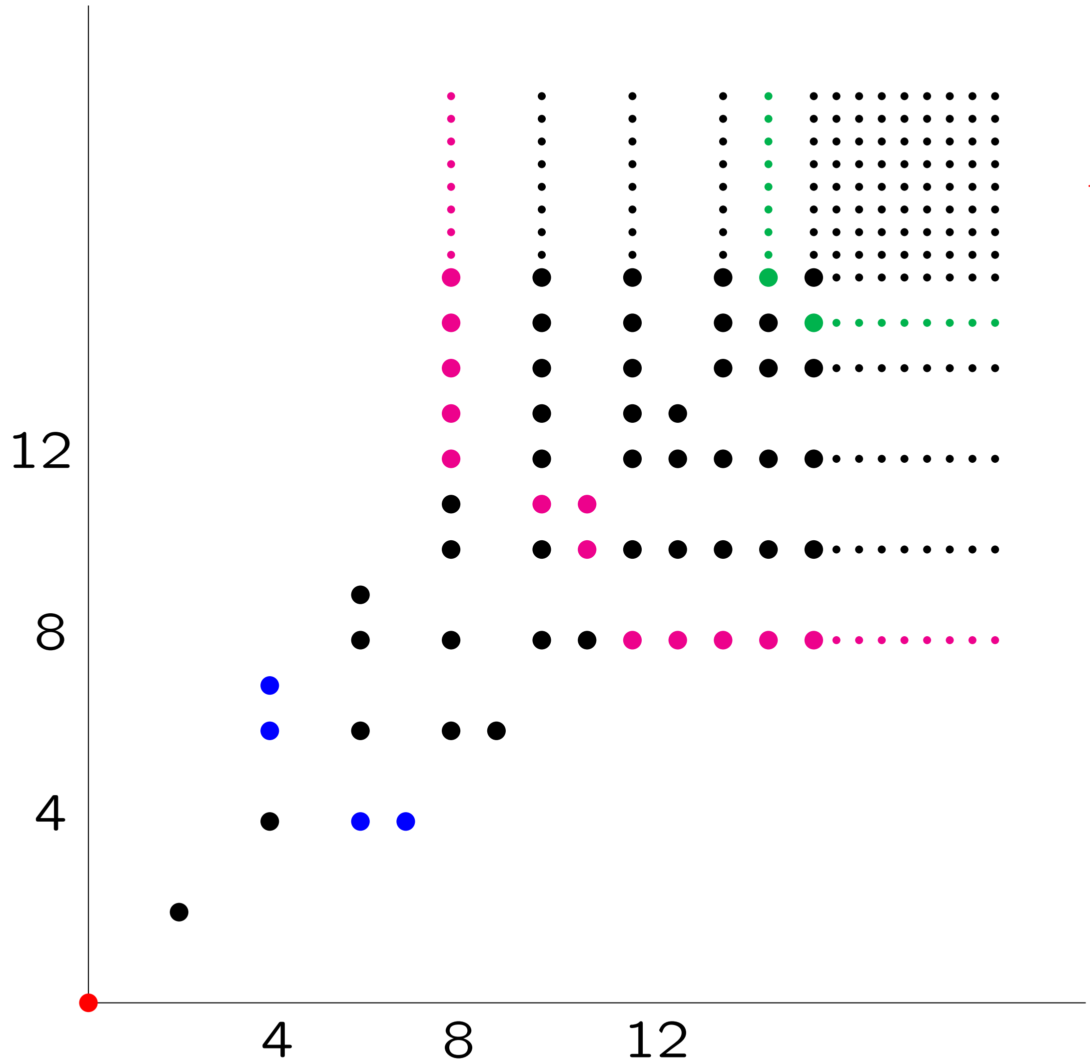
$$A_i = v(T_i)$$

Proposition. Set $R = k[[x]] + k[[x]]y + \cdots + k[[x]]y^{E-1}$, algebroid curve of multiplicity $\boldsymbol{e} = (e_1, \ldots, e_h)$.
If we choose $f = x$ and g as in the previous theorem, then $R_1 = k[[x]] + k[[x]]g + \cdots + k[[x]]g^{E-1}$ and

$$R = k[[x]] + k[[x]]gx + \cdots + k[[x]](gx)^{E-1}$$

Corollary. If $Ap(v(R), \boldsymbol{e}) = \bigcup_{i=0}^{E-1} A_i$, $Ap(v(R_1), \boldsymbol{e}) = \bigcup_{i=0}^{E-1} A'_i$, then for every i, $A_i = A'_i + i\boldsymbol{e}$.
Hence to give a semigroup of a plane curve with $h \geq 2$ branches is equivalent to give its multiplicity tree.

Theorem. [_, D, G, M, M] Characterization of the multiplicity trees of plane curves with any number of branches.
Thus we can give a constructive characterization of all the value semigroup of a plane singularity with any number of branches.



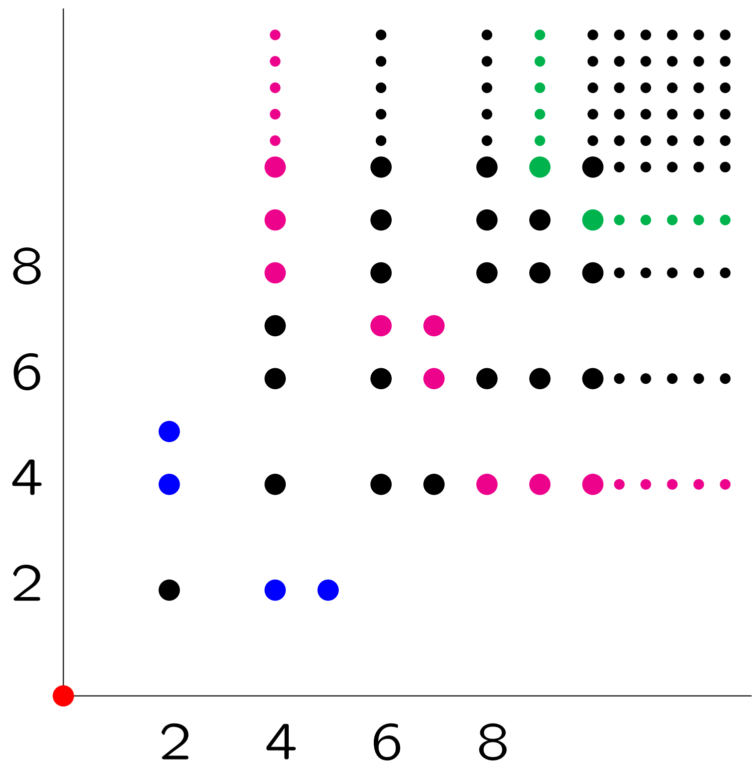
$$R = \frac{k[[x,y]]}{(x^7-y^2)\cap(x^7-x^4+2x^2y-y^2)}$$

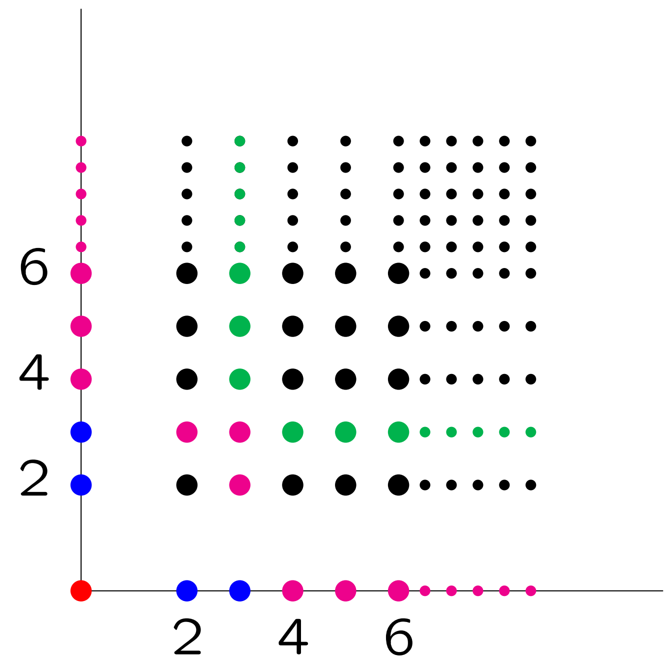
$$x \mapsto (t^2, u^2)$$

$$y \mapsto (t^7, u^4 + u^7)$$

$$\boldsymbol{e} = (2,2)$$

Picture 8. $S = v(R)$ $\quad Ap(S, \boldsymbol{e}) = A_0 \cup A_1 \cup A_3 \cup A_4$





Picture 9. $Ap(v(R_1, \boldsymbol{e})) = A_0 \cup A_1 \cup A_3 \cup A_4$ and $Ap(v(R_2), \boldsymbol{e}_1))$

Now $R_2 \cong R_{2,1} \times R_{2,2}$ is semilocal and $S_2 := v(R_2) = \pi_1(S_2) \times \pi_2(S_2)$.

THANKS FOR YOUR ATTENTION!