On the Set of Betti Elements of a Puiseux Monoid

Scott Chapman

July 8, 2024

Introduction

This talk is based on the paper

On the Set of Betti Elements of a Puiseux Monoid

which has recently appeared in *Bulletin of the Australian Mathematical Society*. It was a product of an MIT sponsored *Primes* project and the co-authors were

Joshua Jang

Jason Mao

Skyler Mao.

Special thanks go to both **Felix Gotti** and **Harold Polo** for seeing this project through to its completion.

What Started It All (For Me)?

Arch. Math., Vol. 61, 521–528 (1993)

0003-889X/93/6106-0521 $ 3.10/0

Factorization in $K[X^2, X^3]$

By

DAVID F. ANDERSON, SCOTT CHAPMAN, FAITH INMAN*) and W. W. SMITH

Introduction. If D is a UFD, then any two factorizations of a nonzero nonunit into the product of irreducible elements have the same length. However, this need not be true for an arbitrary atomic domain (an integral domain is *atomic* if each nonzero nonunit is a product of irreducible elements (atoms)). Following Zaks [19], we say that an atomic domain D is a *half-factorial domain* (HFD) if whenever $x_1 \cdots x_m = y_1 \cdots y_n$ with each $x_i, y_j \in D$ irreducible, then $m = n$. For any field K, the domain $R = K[X^2, X^3] = \{\Sigma a_i X^i \in K[X] \mid a_1 = 0\}$ is probably the simplest example of an atomic domain which is not a HFD since X^2 and X^3 are each irreducible elements of R and $X^6 = X^3 X^3 = X^2 X^2 X^2$. (Clearly R is atomic since R is a (one-dimensional) Noetherian domain. This may also be shown by an easy degree argument.) The domain R is also of interest and has been studied extensively in several other contexts. For example, R is also the simplest example of a nonseminormal domain, and hence $\mathrm{Pic}(R) \neq \mathrm{Pic}(R[T])$ (see [17]).

Recently there has been much activity on HFDs, other factorization properties weaker than unique factorization, and on invariants that measure different lengths of factorizations. In this paper, we compute two such invariants for $R = K[X^2, X^3]$. First, for any

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The elasticity is the most basic of the invariants connected to the study of non-unique factorizations, be in an integral domain (as in the early papers of Anderson, Anderson, Zafrullah), or in a monoid (as in the early papers of Halter-Koch and Geroldinger).

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So a natural question after this paper was, given a numerical monoid S, how do elements factor in

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But shortly after the appearance of this paper, the study of much more involved nonunique factorization constants in an atomic monoid S became popular:

- The Delta Set $\Delta(S)$.
- The Catenary Degree $\mathrm{c}(S)$.
- The Tame Degree $\mathrm{t}(S)$.
- The Omega Function $\omega(x)$.

But to understand handling these constants in $K[X; S]$, **one must have a good understanding of how they work in** S. Problems like this were not seriously approached in affine or numerical monoids until around 2000.

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Betti Elements

Let M be a commutative cancellative monoid. Let $\mathcal{A}(M)$ be its set of atoms. We assume without loss for our discussion that M is reduced (i.e., has a unique unit).

If every element has a factorization into atoms then we call M *atomic*. We let $\mathsf{Z}(M)$ denote the free commutative monoid on the set $\mathcal{A}(M_{\mathrm{red}})$, and the formal sums in $\mathsf{Z}(M)$ are called *factorizations*. Let $\mathsf{Z}(b)$ represent the atomic factorizations of the element b in M.

Betti Elements

The *Betti graph* ∇_b of b is the graph whose set of vertices is $\mathsf{Z}(b)$ having an edge between factorizations $z, z' \in \mathsf{Z}(x)$ precisely when z and z' share an atom. An element of b of M is a *Betti element* if and only if its Betti graph is disconnected. We let $\mathrm{Betti}(M)$ denote the set of Betti elements of M.

We note that Betti elements can be defined in several other equivalent ways using an equivalence relation on $\mathsf{Z}(M)$ or the minimal presentations of M. We focus on this method as it makes the most sense when approaching problems from the direction of factorization theory.

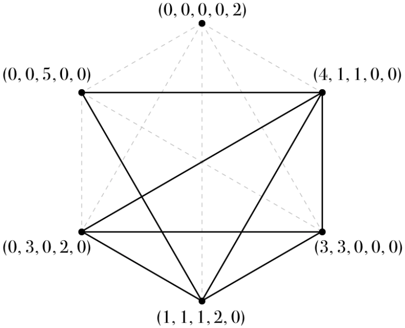

Figure: For $N = \langle 14, 16, 18, 21, 45 \rangle$, the figure shows the Betti graph of $90 \in \text{Betti}(N)$ on the left and that of $84 \notin \text{Betti}(N)$ on the right.

Why Betti Elements?

Betti Elements play a key role in computing several of the constants I reviewed earlier in the setting of a numerical monoid.

Theorem (Chapman, García-Sánchez, Llena - Forum Math. 2009)

If M is a numerical monoid, then

$$c(M) = \max\{c(b) \mid b \in Betti(M)\}.$$

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Hence Key Questions

Questions

Given a commutative cancellative monoid M, what is its complete set of Betti elements (Betti(M))? Can the Betti elements of M be used to draw basic conclusions about the factorization properties of M?

Even in the case where M is a numerical monoid, the first problem is highly computative (it can be done in GAP for affine monoids). There are some exceptional cases.

- $M = \langle n_1, n_2 \rangle$ then $\mathrm{Betti}(M) = \{n_1 n_2\}$.
- $M = \langle n_1, n_2, n_3 \rangle$, then $\mid \mathrm{Betti}(M) \mid = 1$, 2, or 3.
- $M = \langle n_1, n_2, \ldots, n_k \rangle$ then $\mid \mathrm{Betti}(M) \mid \leq \frac{n_1(n_1-1)}{2}$.

Finitely generated monoids have finitely many Betti elements. Also, affine monoids with a unique Betti element are studied in detail in [García-Sánchez, Ojeda, Rosales, J. Algebra Appl. (2013)]

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Puiseux Monoids

Definition

An additive submonoid of M of $\mathbb{Q}$ is a **Puiseux monoid** if it consists of nonnegative rationals.

While Puiseux monoids are natural extensions of numerical monoids, there are many differences.

- Puiseux monoids may not be finitely generated.
- Puiseux monoids may not have the ACCP.
- Puiseux monoids may not be atomic.
- Atomic Puiseux monoids may not have finite elasticity.
- Atomic Puiseux monoids may not be bounded factorization monoids.
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Some Additional Comments

If you want to get deeply into the last six points, then see the following.

[1] F. Gotti and C. O'Neill. The elasticity of Puiseux monoids, *Journal of Commutative Algebra* **12**(2020), 319–331.

[2] S. Chapman, F. Gotti, and M. Gotti, When is a Puiseux monoid atomic?, *The American Mathematical Monthly*, **128**(2021), 302–321.

Comments:

- A finitely generated Puiseux monoid is isomorphic to a numerical monoid.
- If M is a Puiseux monoid and 0 is not a limit point of M, then M is atomic and a bounded factorization monoid.

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A Basic Example

Example

For $r \in \mathbb{Q}_{>0}$, set $r = \frac{\mathsf{n}(r)}{\mathsf{d}(r)}$ where this representation is in lowest terms. We also set

$$S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$$

to be the *cyclic rational semiring* additively generated by the nonnegative powers of r. Obviously, S_r is a Puiseux monoid. Here are some known factorization facts about the S_r monoids.

- S_r is atomic precisely when either $r = 1$ or $\mathsf{n}(r) > 1$.
- If S_r is atomic, then $\Delta(S_r) = \{\mid \mathsf{n}(r) - \mathsf{d}(r) \mid\}$.
- If S_r is atomic and $r \notin \mathbb{N}$, then $\mathsf{c}(r) = \mid \mathsf{n}(r) - \mathsf{d}(r) \mid$ for all $x \in S_r$ with $\mid \mathsf{Z}(x) \mid > 1$. Therefore $\mathsf{c}(S_r) = \mid \mathsf{n}(r) - \mathsf{d}(r) \mid$.

Another Basic Example

Example

Let $\mathbb{P}$ represent the set of positive primes and suppose that $P \subseteq \mathbb{P}$. A *primary Puiseux monoid* is a monoid of the form

$$\langle \frac{a_p}{p} \mid p \in P \text{ and } a_p \in \mathbb{N} \backslash p\mathbb{N} \rangle \subset \mathbb{Q}_{\geq 0}.$$

The factorization properties of primary Puiseux monoids have been studied in detail by F. Gotti and C. O'Neill (*Journal of Commutative Algebra* (2020)). We say that $a \in \mathcal{A}(M)$ is *stable* if the set

$$\{x \in \mathcal{A}(M) | \mathsf{n}(x) = \mathsf{n}(a)\}$$

is infinite, and *unstable* otherwise. Gotti and O'Neill have shown the following.

A Primary Puiseux Monoid Theorem

Theorem

For a primary Puiseux monoid M, the following are equivalent:

1. *M is a FF-monoid;*
2. *M is a BF-monoid;*
3. *Every $a \in \mathcal{A}(M)$ is unstable.*

An Example

Example

Consider the primary Puiseux monoid $M_{\mathbb{P}} := \langle \frac{1}{p} \mid p \in \mathbb{P} \rangle$. It is well known that $M_{\mathbb{P}}$ is atomic with $\mathcal{A}(M_{\mathbb{P}}) = \{\frac{1}{p} \mid p \in \mathbb{P}\}$. It follows a theorem of D.F. Anderson and F. Gotti that every element $q \in M$ can be written uniquely as

$$q = c + \sum_{p \in \mathbb{P}} c_p \frac{1}{p},$$

where $c \in \mathbb{N}_0$ and $c_p \in [\![0, p-1]\!]$ for every $p \in \mathbb{P}$. From this, we can infer that for any element $q \in M$, the conditions $|\mathsf{Z}(q)| = 1$ and $1 \nmid_M q$ are equivalent.

Theorem

$Betti(M_{\mathbb{P}}) = \{1\}$.

The proof is based on the last inference above.

A Theorem on $M_{\mathbb{P}}$

Theorem

$Betti(M_{\mathbb{P}}) = \{1\}$.

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Atomization

Let $(q_n)_{n \geq 1}$ be a sequence consisting of positive rationals, and let $(p_n)_{n \geq 1}$ be a sequence of pairwise distinct primes such that

$$\gcd(p_i, \mathsf{n}(q_i)) = \gcd(p_i, \mathsf{d}(q_j)) = 1$$

for all $i, j \in \mathbb{N}$. Following Gotti and Li (Proc. Amer. Math. Soc. **151** (2023) 2291–2302), we say that

$$M := \left\langle \frac{q_n}{p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is the *Puiseux monoid* of $(q_n)_{n \geq 1}$ *atomized* at $(p_n)_{n \geq 1}$. It is not hard to argue that M is atomic with $\mathcal{A}(M) = \{\frac{q_n}{p_n} \mid n \in \mathbb{N}\}$. It turns out that we can determine the Betti elements of certain Puiseux monoids obtained by atomization.

A Lemma

Lemma

Let M be the Puiseux monoid of $(q_n)_{n\geq 1}$ atomized at $(p_n)_{n\geq 1}$. Then every element $q \in M$ can be uniquely written as follows:

$$q = n_q + \sum_{n\in\mathbb{N}} c_n \frac{q_n}{p_n}, \tag{1}$$

where $n_q \in \langle q_n \mid n \in \mathbb{N}\rangle$ and $c_n \in [\![0, p_n - 1]\!]$ for every $n \in \mathbb{N}$ (here $c_n = 0$ for all but finitely many $n \in \mathbb{N}$).

The Betti Elements of an Atomization

Theorem

Let M be the Puiseux monoid of $(q_n)_{n\geq 1}$ atomized at $(p_n)_{n\geq 1}$. Then the following statements hold.

1. $\mathit{Betti}(M) \subseteq \langle q_n \mid n \in \mathbb{N} \rangle$.
2. $\{q_n \mid n \in \mathbb{N}\} \subseteq \mathit{Betti}(M)$ *if* $\langle q_n \mid n \in \mathbb{N} \rangle$ *is antimatter.*
3. $\mathit{Betti}(M) \subseteq \{q_n \mid n \in \mathbb{N}\}$ *if* $\langle q_n \mid n \in \mathbb{N} \rangle$ *is a valuation monoid.*

A Corollary

Corollary

Let M be the Puiseux monoid of $(q_n)_{n\geq 1}$ atomized at $(p_n)_{n\geq 1}$. If $\langle q_n \mid n \in \mathbb{N}\rangle$ is an antimatter valuation monoid, then

$$Betti(M) = \{q_n \mid n \in \mathbb{N}\}.$$

The Grams Monoid

Example

Let $(p_n)_{n\geq 0}$ be the strictly increasing sequence whose underlying set consists of all odd primes, and consider the Puiseux monoid

$$M := \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\rangle.$$

The monoid M is often referred to as the *Grams' monoid* as it was the crucial ingredient in Grams' construction of the first atomic integral domain not satisfying the ACCP. Observe that M is the atomization of the sequence $(\frac{1}{2^n})_{n\geq 0}$ at the sequence of primes $(p_n)_{n\geq 0}$. Since $\langle \frac{1}{2^n} \mid n \in \mathbb{N}_0 \rangle$ is an antimatter valuation monoid, it follows from the last Corollary that

$$\mathrm{Betti}(M) = \left\{ \frac{1}{2^n} \;\middle|\; n \in \mathbb{N}_0 \right\}.$$

Betti Cardinalities in Puiseux Monoids

Proposition

For each $b \in \mathbb{N} \cup \{\infty\}$, there exists an atomic Puiseux monoid M such that $|\mathrm{Betti}(M)| = b$.

A Sketch of the Proof

Proof.

The Grams' monoid has infinitely many Betti elements. Fix $b \in \mathbb{N}$. Now consider the sequence $(q_n)_{n\geq 1}$ whose terms are defined as $q_{kb+r} := r+1$ for every $k \in \mathbb{N}_0$ and $r \in [\![0, b-1]\!]$. Now let $(p_n)_{n\geq 1}$ be a strictly increasing sequence of primes such that $p_n > b$ for every $n \in \mathbb{N}$. Then $\gcd(p_i, \mathsf{n}(q_i)) = \gcd(p_i, \mathsf{d}(q_j)) = 1$ for all $i, j \in \mathbb{N}$. Let M be the Puiseux monoid we obtain after atomizing the sequence $(q_n)_{\geq 1}$ at the sequence $(p_n)_{n\geq 1}$. It follows from part (4) of our previous theorem that

$$\mathrm{Betti}(M) \subseteq \{q_n \mid n \in \mathbb{N}\} = [\![1, b]\!].$$

Using part (1) of the same theorem, we obtain

$$[\![1, b]\!] \subseteq \mathrm{Betti}(M)$$

and the result follows. □

Example

Let r be a non-integer positive rational, and we return to the Puiseux monoid $S_r := \langle q^n \mid n \in \mathbb{N}_0 \rangle$. It is well known that S_r is atomic (provided that $q^{-1} \notin \mathbb{N}$) and $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$. It follows from our main theorem that

$$\mathsf{Betti}(S_r) = \{\mathsf{n}(r) r^n \mid n \in \mathbb{N}_0\}.$$

Thus, S_r is an atomic Puiseux monoid with infinitely many Betti elements. When $r > 1$, it follows that S_r is an FFM (in particular S_r satisfies the ACCP).