ON REFLEXIVE IDEALS

In the last years some classes of ideals, like e.g. reflexive ideals and trace ideals, were studied in the context of one-dimensional rings (see [4], [5], [6]). Since several properties of one-dimensional rings can be translated for numerical semigroups and, often, results proved for numerical semigroups can be generalized for one-dimensional rings, it is natural to deepen the study of these classes of ideals in the numerical semigroup context. The aim of this talk is to present some new results for reflexive ideals of numerical semigroups, contained in a joint work in progress with M. D'Anna and F. Strazzanti.

A **numerical semigroup** is a set $S \subseteq \mathbb{N}$ such that $0 \in S$, S is closed under the usual $+$ and $\mathbb{N} \setminus S$ is finite; a **relative ideal** of S is a set $E \subseteq \mathbb{Z}$ for which there exists $s \in S$ such that $s + E \subseteq S$ and $E + S \subseteq E$ and a relative ideal is said to be **proper** if it is contained in the numerical semigroup. This definition of relative ideal mimic the definition of fractional ideal of a ring.

Since S itself is a proper ideal and every ideal has finite complement with $\mathbb{N}$ and minimum in $\mathbb{Z}$, we can consider $\mathcal{F}(E) := \max(\mathbb{N} \setminus E)$, $c(E) := \mathcal{F}(E) + 1$ and $m(E) := \min(E)$ which are defined to be respectively the **Frobenius number**, the **conductor** and the **multiplicity** of E (in this context $m(S) = 0$). We can therefore represent S as

$$S = \{s_0 = 0, s_1, \dots, s_n = c, \rightarrow\}.$$

The sets $M(S) := S \setminus \{0\}$, $K(S) := \{x \in \mathbb{Z} \ : \ \mathcal{F}(S) - x \notin S\}$ and $C(S) := \{x \ : \ x \geq c\}$ are ideals, called **maxima ideal**, **standard canonical ideal** and **conductor ideal** respectively. An ideal E is said to be **integrally closed** if there exists $s \in S$ such that $E = \{t \in S : t \geq s\}$, and in this case we denote it as $E = S(s)$. In particular if $s = s_i$ for some $i \in \mathbb{N}$ we write $S(s) = S_i$.

Definition 1. *A relative ideal E of S is said to be **reflexive** if*

$$E = S - (S - E).$$

Definition 2. *A numerical semigroup S is said to be **Arf** if for every $s, t, u \in S$, such that $s \leq t \leq u$, then $u + t - s \in S$.*

Definition 3. *A numerical semigroup S is said to be **almost-symmetric** if $K(S) \subseteq M(S) - M(S)$.*

The first main result about reflexive ideals for numerical semigroups is the following:

Theorem 1. *Let S be a numerical semigroup, E a relative ideal of S and $\tilde{E} := E + \mathcal{F}(S) - \mathcal{F}(E)$ with $s_i = \min(\tilde{E})$. Then the following conditions are equivalent:*

(1) *E is reflexive;*
(2) *$\tilde{E} \subseteq S$ and E is S_i−reflexive;*
(3) *$\tilde{E} \subseteq S$ and there exists $j \leq i$ such that E is S_j−reflexive;*
(4) *$\tilde{E} \subseteq S$ and E is S_j−reflexive for all $j \leq i$;*

and deduce that

Corollary 2. *With the same notation of Theorem 1, if E a reflexive ideal of S; then*

$$(S_i - S_i) + s_i \subseteq \tilde{E} \subseteq S_i;$$

which easily implies

Corollary 3. *The following conditions for a numerical semigroup S are equivalent:*
(1) *S is Arf;*
(2) *an ideal of S is reflexive if and only if it is integrally closed.*

For almost-symmetric we have the following

Proposition 4. *The following conditions are equivalent*
(1) *S is almost-symmetric;*
(2) *a non principal relative ideal E of S is reflexive if and only if $K \subseteq E - E$.*

Many results can be generalized for Arf and almost-Gorenstein rings assuming that $\overline{R}$, the integral closure of R in its total ring of fractions, is a finite $R-$module. In general the task of generalization is not trivial: for instance integrally closed ideals of a numerical semigroup are clearly linearly ordered by inclusion, but this is not true, in general.

In this context, a local one-dimensional reduced ring $(R, \mathfrak{m})$ with finite residue field is said to be **Arf** if every integrally closed ideal of R is stable, i.e. if for all $x, y, z \in R$ with x non-zero divisor such that $\frac{y}{x}, \frac{z}{x} \in \overline{R}$ one has that $\frac{yz}{x} \in R$ (see [7]).

The corresponding result of Corollary 3 for Arf rings is then

Theorem 5. *The ring R is Arf if and only if every reflexive ideal I such that $C \subseteq I \subseteq R$ is integrally closed.*

A local one-dimensional ring $(R, \mathfrak{m})$ is **almost Gorenstein** if $\omega_R \subseteq \mathfrak{m} : \mathfrak{m}$, where ω_R is a canonical module of R such that such that $R \subseteq \omega_R \subseteq \overline{R}$ (see [1]). We can translate Proposition 4 as

Proposition 6. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring and assume that there exists a canonical module ω_R of R such that $R \subseteq \omega_R \subseteq \bar{R}$. The following statements are equivalent:*
(1) *R is almost Gorenstein;*
(2) *A non-principal fractional ideal I of R is reflexive if and only if $\omega_R \subseteq I : I$;*
(3) *A non-principal ideal I of R is reflexive if and only if $\omega_R \subseteq I : I$.*

References

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