Some results on Wilf's conjecture

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Wilf's conjecture (1978) states that for any numerical semigroup S of conductor c, one has $|P||L| \geq c$, where P is the set of primitive elements (i.e. minimal generators) of S and $L = S \cap [0, c-1]$. Let $S^* = S \setminus \{0\}$ and $m = \min(S^*)$, the multiplicity of S. Some currently open cases in Wilf's conjecture are

$$|P| \geq 4, \quad \text{or } |L| \geq 13, \quad \text{or } m \geq 20, \quad \text{or } c > 3m, \quad \text{or } |P| < m/3.$$

Let $D = S^* + S^*$ be the set of decomposable elements of S. Thus $S = \{0\} \sqcup P \sqcup D$. Let $A = S \setminus (m + S)$ be the Apéry set of S with respect to m. It turns out that the larger the proportion $|A \cap D|/m$ is, the more difficult it is to prove Wilf's conjecture for S. Thus, understanding the fine structure of $A \cap D$ is key to making progress towards Wilf's conjecture.

In this talk, we shall outline a recently submitted proof of Wilf's conjecture in the case $|P| \geq m/4$ under the simplifying hypothesis $c \in m\mathbb{N}$. The proof relies on *divsets*, i.e. finite subsets X of monomials in commuting variables $x_1, \ldots, x_n$ which are closed under taking divisors. We use divsets as abstract multiplicative models of $A \cap D$.

We associate to a divset X a suitable graph $G(X)$, whose edges are all pairs $\{u, v\} \subseteq X$ such that $uv \in X$. Let us denote by $\mathrm{vm}(X)$ the *vertex-maximal matching number* of $G(X)$. It turns out that for the case under study, namely

$$|P| \geq m/4 \ \& \ c \in m\mathbb{N},$$

it suffices to consider divsets X such that $\mathrm{vm}(X) \leq 5$. Finding and analyzing all such divsets allows one to reach the desired conclusion, namely that Wilf's conjecture holds in that case.

As an application, in a separate paper with coauthors M. Delgado and J. Fromentin, we verified Wilf's conjecture for all numerical semigroups S of genus $g \leq 120$ such that $c \in m\mathbb{N}$, by computational means and a suitable trimming of the numerical semigroups tree.