COUNTING NUMERICAL SEMIGROUPS BY THEIR MAXIMUM PRIMITIVE AND THE ASYMPTOTIC WILF'S CONJECTURE

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Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$. A numerical semigroup S is a cofinite additive submonoid of $\mathbb{N}$, i.e. S is a subset of $\mathbb{N}$ which is closed under addition, contains 0, and has a finite complement. Equivalently, we can characterise a numerical semigroup as the submonoid generated by a subset of $\mathbb{N}$ which has greatest common divisor equal to 1. Let $\mathcal{S}$ be the set of all numerical semigroups. The various properties of numerical semigroups can be viewed as maps from $\mathcal{S}$ to other sets.

Every numerical semigroup is generated by some of its subsets, and the unique subset minimal under inclusion among the set of all generating subsets is known as the set of *minimal generators* of S or the set of *primitives* of S, and it is denoted by $\mathrm{P}(S)$. One can verify that this property describes the map $\mathrm{P} : \mathcal{S} \to 2^{\mathbb{N}}$ defined by

$$\mathrm{P}(S) = S \setminus (S^* + S^*)$$

where $S^* = S \setminus \{0\}$. The elements of $\mathrm{P}(S)$ are called the primitive elements of S. The cardinality of $\mathrm{P}(S)$, known as the embedding dimension and denoted $e(S)$, is another important property which defines the map $e : \mathcal{S} \to \mathbb{N}$, i.e.

$$e(S) = |\,\mathrm{P}(S)|.$$

Similarly, we have the Frobenius map $\mathrm{F} : \mathcal{S} \to \mathbb{N} \cup \{-1\}$ defined by

$$\mathrm{F}(S) = \max(\mathbb{Z} \setminus S)$$

where $\mathrm{F}(S)$ is known as the *Frobenius number* of S. It is a famous open problem, known as the Frobenius coin-exchange problem or the Frobenius problem, to determine a formula for $\mathrm{F}(S)$ in terms of $\mathrm{P}(S)$. Let us define

$$\mathcal{N}_f = \mathrm{F}^{-1}(f) = \{S \in \mathcal{S} : \mathrm{F}(S) = f\}$$

and denote its cardinality by N_f. A kind of well studied inverse problem to the Frobenius problem asks us to find N_f for any given f. In other words, it asks us to compute the sequence

$$(N_f) = \{|\,\mathrm{F}^{-1}(f)| : f \in \mathbb{N} \cup \{-1\}\}.$$

This problem is also known as the problem of *counting numerical semigroups by the Frobenius number* or *counting by Frobenius number*, in short. Note that since the elements in the complement of $S (\neq \mathbb{N})$ are called the gaps of S, alternatively one may call this problem as

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counting numerical semigroups by the maximum gap. Several other ways of counting numerical semigroups exist: counting by *genus*, box counting, etc.

Similar to the counting by the maximum gap, we define a way of counting based on the maximum of the primitive elements. Let $\mu : \mathcal{S} \to \mathbb{N}$ be defined as

$$\mu(S) = \max \mathrm{P}(S)$$

where $\mu(S)$ is referred to as the maximum primitive of S. Given a positive integer n, we consider the set

$$\mathcal{A}_n = \mu^{-1}(n) = \{S \in \mathcal{S} : \max \mathrm{P}(S) = n\}$$

and denote its cardinality by A_n. Note that A_n is finite for every n, since every numerical semigroup in $\mathcal{A}_n$ has its set of minimal generators contained in the finite interval $[1, n] \cap \mathbb{N}$. The *counting by maximum primitive* will refer to the problem of determining the number of numerical semigroups which have a given number n as their maximum primitive. In other words, it asks us to compute the sequence

$$(A_n) = \{|\mu^{-1}(n)| : n \in \mathbb{N}\}.$$

We show that the counting by maximum primitive is related to the counting by Frobenius number through a map defined as follows:

Definition 1. Given a positive integer f, we define the map $\Phi : \mathcal{A}_f \to \mathcal{N}_f$ by

$$\Phi(S) = (S \setminus \{f\}) \cup (f, \infty).$$

This map is injective, and thus we have that $N_f \geq A_f$. Furthermore, some arguments based on this map yield the following equation:

Theorem 2.

$$N_f = \sum_{d|f} A_{\frac{f}{d}}.$$

Thus the sequence (N_f) can be calculated from the sequence (A_f) and, in other words, the problem of Frobenius counting can be solved through a solution to the problem of maximum primitive counting.

Corollary 3. *Thus for any f,*

$$N_f \geq A_f + 1.$$

And, for p prime,

$$N_p = A_p + 1.$$

The Mobius inversion theorem has the following implication:

Theorem 4. *Let $n > 2$. Then*

$$A_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) \cdot N_d$$

where $\mu(k)$ denotes the sum of all k-th roots of unity.

Thus the sequences (N_f) and (A_f) can be obtained from one another, and thus the Frobenius counting is equivalent to the maximum primitive counting in this sense.

We also study the growth of the sequence (A_f), and show the following result related to the monotonicity of the odd and even subsequences of (A_n).

Theorem 5. *For $f > 1$,*

$$A_{f+2} > A_f.$$

Next we look at the asymptotic version of a well studied open problem known as Wilf's conjecture. Given a numerical semigroup S, let $L(S)$ denote the set $S \cap [0, \mathrm{F}(S))$. Wilf's conjecture states that for every numerical semigroup S, the following inequality holds

$$|\,\mathrm{P}(S)| \cdot |L(S)| \geq \mathrm{F}(S) + 1.$$

We show that Wilf's conjecture is true for almost every semigroup under counting by the maximum primitive in the following sense:

Theorem 6.

$$\lim_{n\to\infty} \frac{|\{S \in \mathcal{A}_n : |\,\mathrm{P}(S)| \cdot |L(S)| \geq \mathrm{F}(S) + 1\}|}{A_n} = 1.$$

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