The value semigroup of a plane curve singularity with several branches

MARCO D'ANNA

A classical tool to get numerical invariants of a curve singularity is the study of its value semigroup (see, e.g. [16], [12], [5], [10], [11], [6], [15]). In case of a one branch singularity this semigroup is a numerical semigroup (i.e. a submonoid of $\mathbb{N}$ with finite complement in it); in case the singularity has h branches, this semigroup is a subsemigroup of $\mathbb{N}^h$, belonging to the class of the so-called "good semigroups". Despite their name, the combinatoric of good semigroups is quite problematic; moreover, for $h \geq 2$, it is an open problem to understand which good semigroups can be realized as value semigroups ([2]). In case of a plane singularity with one branch, an old result of Apéry ([1]) shows that there is a particularly strict connection between the value semigroups of the singularity and of its blowup; this connection is obtained using a particular set of generators of the semigroup, named "Apéry set". Using that result, it is possible to show very easily the classical fact that the equisingularity classes determined by the multiplicity sequences coincide with the equisingularity classes determined by the value semigroups ([5]). In particular, this method allows to easily reconstruct the numerical semigroup associated to a plane branch singularity starting from the multiplicity sequence and vice-versa ([3]).
When the singularity has more than one branch, in order to generalize the Apéry result, two main problems arise: firstly, the Apéry set becomes an infinite set; secondly, in the process of blowing-up it is necessary to deal with semilocal rings, that cannot be presented as quotients of a power series ring in two variables, as it happens in the local case. These problems where partially solved twenty years ago in [4] for the two branches case, with the help of a result of Garcia ([12]), that holds only for a two-branches singularity.

In this talk, after describing some key definitions and results on good semigroups and value semigroups of a curve singularity with h branches, I will recall the Apéry process for a plane branch and then, I will present some recent results obtained in a joint project with F. Delgado de la Mata, L. Guerrieri, N. Maugeri and V. Micale ([7]), that give a complete solution of the above metioned problems and lead to a constructive characterization of the value semigroups of any plane curve singularity.

More precisely, we use the fact that the Apéry set of a good semigroup can be seen as a union of substets, pairwise disjoints, that we call levels ([8], [9], [13], [14]). When we start form a curve singularity or from one of its blow-ups (so the ring R is not necessarily local), we can describe the ring we are dealing with as

$$R = K[[F]] + K[[F]]G + \cdots + K[[F]]G^{N-1},$$

where $F \in R$ is any element of a fixed value, N is the sum of the components of the value of F, and G can be properly determined. Setting $R_i = K[[F]]+K[[F]]G+\cdots+K[[F]]G^i$, we show that the levels of the Apéry set of R can be described as the following value sets:

$$v(T_i) = v(\{G^i + \phi \mid \phi \in R_{i-1} \text{ and } v(G^i + \phi) \notin v(R_{i-1})\}).$$

Using this theorems, it is possible to prove the following

Theorem 1. *Let $\mathcal{O}$ be the ring of a plane algebroid curve and suppose its blow-up ring $\mathcal{B}(\mathcal{O})$ to be not local. Let ω be the minimal nonzero element of $\mathcal{O}$. Let A_i and A_i' denote the i-th levels of the Apéry sets with respect to ω of $v(\mathcal{O})$ and of $v(\mathcal{B}(\mathcal{O}))$, respectively. Then $A_i = A_i' + i\omega$.*

The last part of our work is devoted to give an explicit characterization of the admissible multiplicity trees of a plane curve singularity. This characterization, together with Theorem 1, allows to contruct all the possible value semigroup of a plane singularity with any number of branches.
Notice that the characterization of admissible multiplicity trees as well as the results about the Apéry set are independent of the characterisitc of the base field (that we only require to be infinite).

References

[1] R. Apéry. *Sur les branches superlinéaires des courbes algébriques.* C.R. Acad. Sci. Paris, (222):1198-1200, 1946.

[2] V. Barucci, M. D'Anna, R. Fröberg, *Analytically unramified one-dimensional semilocal rings and their value semigroups*, J. Pure Appl. Alg. **147** (2000), 215-254.

[3] V. Barucci, M. D'Anna, R. Fröberg, *On plane algebroid curves,* in Commutative Ring theory and applications, 37–50, Lecture Notes in pure and applied mathematics, vol. 231, Marcel Dekker 2002.

[4] V. Barucci, M. D'Anna, R. Fröberg, *The Apéry algorithm for a plane singularity with two branches*, Beitrage zur Algebra und Geometrie, 46 (1), 2005;

[5] A. Campillo, *Algebroid curves in positive characteristic*, LNM Vol. **813**. Springer-Verlag, Heidelberg-Berlin-New York (1980).

[6] A. Campillo, F. Delgado, S. M. Gusein-Zade, *On generators of the semigroup of a plane curve singularity*, J. London Math. Soc. (2) **60** (1999), 420-430.

[7] M. D'Anna, F. Delgado, L. Guerrieri, N. Maugeri, V. Micale *The value semigroup of a plane curve singularity with several branches*, preprint (2024).

[8] M. D'Anna, L. Guerrieri, V. Micale, *The Apéry Set of a Good Semigroup*. Advances in Rings, Modules and Factorizations, Springer Proc. Math. Stat., 321 (2020), p. 79-104.

[9] M. D'Anna, L. Guerrieri, V. Micale, *The type of a good semigroup and the almost symmetric condition*. (2020) Mediterr. J. Math. 17, no.1, Paper No. 28, 23pp.

[10] F. Delgado, *The semigroup of values of a curve singularity with several branches*, Manuscripta Math. **59** (1987), 347-374.

[11] F. Delgado, *Gorenstein curves and symmetry of the semigroup of value*, Manuscripta Math. **61** (1988), 285-296. Perspectives, Springer.

[12] A. García, *Semigroups associated to singular points of plane curves*, J. Reine Angew. Math. **336** (1982), 165-184.

[13] L. Guerrieri, N. Maugeri, V. Micale, *Partition of the complement of good semigroup ideals and Apéry sets*, Communications in Algebra, 49, No. 10, 4136-4158 (2021).

[14] L. Guerrieri, N. Maugeri, V. Micale, *Properties and applications of the Apéry set of good semigroups in* $\mathbb{N}^d$, Journal of Algebraic Combinatorics, https://doi.org/10.1007/s10801-022-01185-4 (2022).

[15] R. Waldi. *On the equivalence of plane curve singularities,* Communications in Algebra, 28(9) (2000), 4389-4401.

[16] O. Zariski, *Le problème des modules pour les branches planes,* Hermann, Paris, 1986

DIPARTIMENTO DI MATEMATICA E INFORMATICA - UNIVERSITY OF CATANIA
V.LE A. DORIA 6, 95125, CATANIA, ITALY
E-mail address: marco.danna@unict.it