A LINEAR VARIANT OF NEARLY GORENSTEINNESS AND PROJECTIVE MONOMIAL CURVES

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Abstract. Any 2-dimensional standard graded affine semigroup ring is isomorphic to what is called a *projective monomial curve*, which is a good class that can be analyzed using techniques of numerical semigroups. In commutative ring theory, nearly Gorensteinness is one of several generalizations of the Gorenstein property. In this talk, we characterize non-Gorenstein and nearly Gorenstein projective monomial curves of codimension 2 and 3. Moreover, we introduce the condition (♮) and discuss this condition in projective monomial curves. This is a condition broader than nearly Gorensteinness defined in (semi) standard graded rings, and it is a concept sharing good properties in commutative algebra which are similar to ones nearly Gorenstein rings satisfy.

1. Introduction

Let $\Bbbk$ be a field, and let us denote the set of nonnegative integers by $\mathbb{N}$. Cohen–Macaulay rings and Gorenstein rings are important types of rings and play a crucial role in the theory of commutative algebras. Many kinds of rings are defined for studying new classes of local or graded rings which are Cohen–Macaulay but not Gorenstein. For example, there are *nearly Gorenstein* rings, *almost Gorenstein* rings and *level* rings, and so on. Let R be a Cohen–Macaulay $\mathbb{N}$-graded $\Bbbk$-algebra with canonical module ω_R. According to [3], R is called nearly Gorenstein if $\mathrm{tr}(\omega_R) \supset \mathbf{m}$. Here, $\mathrm{tr}(\omega_R)$ is the ideal generated by the image of ω_R through all homomorphism of R-modules into R if and only if $\mathrm{tr}(\omega_R) = \mathbf{m}$ (see [3, Definition 2.2]). On the other hand, according to [8], R is called level if all the degrees of the minimal generators of ω_R are the same. There are some results about the characterization of nearly Gorenstein rings on some concrete graded rings related to numerical semigroups. For example, a full characterization is known for numerical semigroup rings with small embedding dimension ([2]), and nearly Gorenstein projective monomial curves of codimension 2 and 3 ([6]). Regarding the latter, we will explain the details. Let $\mathbb{P}_\Bbbk$ be a projective space over $\Bbbk$ and let C be a monomial curve in $\mathbb{P}^n_\Bbbk$. Moreover, let $\mathbb{A}(C)$ be its projective coordinate ring. $\mathbb{A}(C)$ is isomorphic to the affine semigroup ring $\Bbbk[S_\mathbf{a}] = \Bbbk[t^{a_n}, s^{a_1}t^{a_n-a_1}, s^{a_2}t^{a_n-a_2}, \cdots, s^{a_{n-1}}t^{a_n-a_{n-1}}, s^{a_n}]$, where

$$S_\mathbf{a} = \langle (0,a_n),(a_1,a_n-a_1),(a_2,a_n-a_2),\cdots,(a_{n-1},a_n-a_{n-1}),(a_n,0)\rangle.$$

We can assume $0 < a_1 < \cdots < a_n$ are integers with $\gcd(a_1,a_2,\cdots,a_n) = 1$. We call this semigroup $S_\mathbf{a}$ the projective monomial curve defined by $\mathbf{a}$. We call $S_\mathbf{a}$ is Cohen–Macaulay, Gorenstein, nearly Gorenstein and level if its affine semigroup ring $\Bbbk[S_\mathbf{a}]$ is so, respectively. Then the following is true.

Theorem 1.1 ([6, Theorem A]). *Let $S_\mathbf{a}$ be the non-Gorenstein and Cohen–Macaulay projective monomial curve defined by $\mathbf{a} = a_1,\cdots,a_n$ with $0 < a_1 < \cdots < a_n$ and $\gcd(a_1,a_2,\cdots,a_n) = 1$.*
(a) If $n = 3$, then $S_\mathbf{a}$ is nearly Gorenstein $\Leftrightarrow$ $\mathbf{a} = k, k+1, 2k+1$ for some $k \geq 1$.
(b) If $n = 4$, then $S_\mathbf{a}$ is nearly Gorenstein $\Leftrightarrow$ $\mathbf{a} = 1,2,3,4$ or $S_\mathbf{a} \cong S_{2k-1,2k+1,4k,6k+1}$ for some $k \geq 1$.
Moreover, every nearly Gorenstein projective monomial curve of codimension at most 3 is level.

Remark 1.2. Note that $\Bbbk[S_\mathbf{a}]$ is always Gorenstein if $n = 2$ because it is a hypersurface.

Remark 1.3. It is known that there exists a nearly Gorenstein but not level projective monomial curve of codimension 4. Indeed, $S_{4,9,12,13,21}$ is such a example (see [5, Theorem 3.11]).

Thus, nearly Gorenstein rings that are not Gorenstein may lack concrete examples because they are too close to being Gorenstein. The main purpose of this talk is to introduce a property that is broader than nearly Gorensteinness in semi-standard graded rings. Semi-standard graded rings are a generalization of standard graded rings and include important classes of combinatorial commutative algebras, such as *Ehrhart rings* arising from polytopes and *face rings* arising from simplicial posets. Of course, projective monomial curves are standard graded rings, so they can also be considered as semi-standard graded rings. For a graded R-module M, we denote the part of M of degree i by M_i.

Definition 1.4. Let R be a semi-standard graded Cohen–Macaulay $\Bbbk$-algebra with a unique graded maximal ideal $\mathbf{m}$. We call R satisfies condition ($\natural$) if $(\mathrm{tr}_R(\omega_R))_1 R \supset \mathbf{m}^k$ for some $k > 0$.

Note that the following hierarchy is correct in semi-standard graded Cohen–Macaulay rings:

R is nearly Gorenstein $\Rightarrow$ R satisfies ($\natural$) $\Rightarrow$ R is Gorenstein on the punctured spectrum.

Note that R is Gorenstein on the punctured spectrum if and only if $\mathrm{tr}(\omega_R) \supset \mathbf{m}^k$ for some $k > 0$ (see [3, Lemma 2.1]). In [1], a property broader than nearly Gorenstein, namely that a polytope P has a Minkowski decomposition $P = P + \lfloor P \rfloor$, is discussed. Furthermore, if we consider semi-standard graded affine semigroup rings $\Bbbk[S]$, the authors discover that the condition $\mathrm{tr}(\omega_R) \supset (x^{\mathbf{e}} : \mathbf{e} \in E_S)$ satisfies good properties of nearly Gorenstein rings with respect to h-vectors ([7, Corollary 3.6]), where E_S denotes a subset of generators of S corresponding to 1-dimensional faces of the cone $\mathbb{R}_{\geq 0} S$. In fact, these conditions are equivalent to ($\natural$) in Ehrhart rings, semi-standard graded rings, or similar settings.

Proposition 1.5. *Let $R = \Bbbk[S]$ be a Cohen–Macaulay semi-standard graded affine semigroup ring and let E_S be a subset of generators of S corresponding to 1-dimensional faces of the cone $\mathbb{R}_{\geq 0} S$. Then R satisfies ($\natural$) if and only if $tr(\omega_R) \supset (x^{\mathbf{e}} : \mathbf{e} \in E_S)$. In particular, when $S \cong S_{\mathbf{a}}$ is a Cohen–Macaulay projective monomial curve, then R satisfies ($\natural$) if and only if $tr(\omega_R) \supset (s^{a_n}, t^{a_n})R$.*

In codimension 2, there are no non-nearly Gorenstein projective monomial curves satisfying ($\natural$).

Proposition 1.6. *Let $S_{\mathbf{a}}$ be the Cohen–Macaulay projective monomial curves defined by $\mathbf{a} = a_1, a_2, a_3$. Then $S_{\mathbf{a}}$ is nearly Gorenstein if and only if $R = \Bbbk[S_{\mathbf{a}}]$ satisfies ($\natural$).*

In codimension 3 or higher, we can construct non-nearly Gorenstein projective monomial curves satisfying condition ($\natural$). If R is Gorenstein on the punctured spectrum, then $R/\mathrm{tr}(\omega_R)$ is Artinian and its length $\ell(R/\mathrm{tr}(\omega_R))$ as an R-module is finite. This is called the *residue* of R. Note that R is nearly Gorenstein if and only if R is Gorenstein on the punctured spectrum with $\ell(R/\mathrm{tr}(\omega_R)) = 1$. The cardinality of minimal generating system of ω_R is called *Cohen–Macaulay type* of R. We denote it by $r(R)$.

Example 1.7. For any integer $k, r \geq 2$, put $\mathbf{a} = 1, k, 2k, \cdots, (r+1)k$. Then $R = \Bbbk[S_{\mathbf{a}}]$ satisfies ($\natural$) with $(\ell(R/\mathrm{tr}(\omega_R)), r(R)) = (k-1, r)$.

Example 1.8. Put $\mathbf{a} = 2, 5, 6, 11$. Then $R = \mathbb{Q}[S_{\mathbf{a}}]$ is non-level projective monomial curves satisfies ($\natural$) with both of codimension is 3. In other words, considering this extended condition ($\natural$), non-level codimension 3 projective monomial curve appears.

In this talk, we discuss condition ($\natural$) in the context of projective monomial curves and present the results obtained.

References

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