TANGENT CONES OF CERTAIN MONOMIAL CURVES

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ABSTRACT. The Cohen- Macaulayness of the tangent cone and behaviour of the Hilbert function of a ring are classical topics in local algebra. We study the tangent cone for a family of numerical semigroups generated by concatenation of arithmetic sequences. We give the explicit structure of tangent cones of aforementioned family of numerical semigroups with some other classical curves. It is a very interesting observation that the tangent cones for all the classes of numerical semigroups except symmetric class are Cohen-Macaulay.

1. INTRODUCTION

It is well known that if the tangent cone is Cohen Macaulay then the Hilbert function of the local ring associated to the monomial curve is non decreasing. In 1978[7], J.D.Sally stated the following conjecture, *The Hilbert function of a one dimensional Cohen Macaulay local ring with small enough(say at most three?) embedding dimension, is non-decreasing*. In 1993[8], Juan Elias proved sally's conjecture for the one dimensional Cohen Macaulay ring of embedding dimension three in the equi-characteristic case.

In general, by the examples given in [9], [11] and [13], it can be seen for higher embedding dimension this conjecture is not true. However, there are no examples of numerical semigroup rings of $4 \leq e$(embedding dimension) ≤ 9 with decreasing Hilbert function. In 2009[10], Rossi posed the following conjecture,

If R is a Gorenstein one-dimensional local ring, is it true that the Hilbert function of the ring is not decreasing? It is equivalent to ask whether the Hilbert function of a semigroup ring corresponding to the symmetric numerical semigroup, is non-decreasing.

In [2] it has been shown that the Hilbert function for every Gorenstein semigroup ring with m(multiplicity) $\leq e+4$ is non decreasing. Recently in [1], it has been shown that there are families of monomial curves which gives negative answer to this problem. However this problem is still open for the Gorenstein local ring associated to symmetric numerical semigroup ring with $4 \leq e \leq 9$.

In [12] a very important family of numerical semigroups (*Numerical Semigroups generated by concatenation of arithmetic sequences*) is defined by the authors. This family of neumerical semigroups has three classes *Unbounded, Symmetric* and *almost maximal*. All three classes exhibited diverse behaviour and the tangent cone is likely to throw some more light on such diverse nature of the classes. This paper will include the study of the tangent cones of these mentioned classes with respect to the conjectures mentioned above. Some

2010 *Mathematics Subject Classification.* Primary 13C40, 13P10.

Key words and phrases. Numerical semigroups, Symmetric numerical semigroups, Apéry set, Monomial curves.

more numerical semigroups, which have been found in the literature would be considered as well for a better understanding of these questions.

2. Numerical semigroups and Concatenation

Let Γ be a numerical semigroup. It is true that the set $\mathbb{N}\backslash\Gamma$ is finite and that the semigroup Γ has a unique minimal system of generators $a_1 < \cdots < a_e$. The greatest integer not belonging to Γ is called the *Frobenius number* of Γ, denoted by $F(\Gamma)$. The integers a_1 and e are known as the *multiplicity* and the *embedding dimension* of the semigroup Γ, usually denoted by $m(\Gamma)$ and $e(\Gamma)$ respectively. The *Apéry set* of Γ with respect to a non-zero $a \in \Gamma$ is defined to be the set $\mathrm{Ap}(\Gamma, \mathrm{a}) = \{\mathrm{s} \in \Gamma \mid \mathrm{s} - \mathrm{a} \notin \Gamma\}$.

Given integers $a_1 < \cdots < a_e$; the map $\nu : k[x_1, \ldots, x_e] \longrightarrow k[t]$ defined as $\nu(x_i) = t^{a_i}$, $1 \leq i \leq e$, defines a parametrization for an affine monomial curve; the ideal $\ker(\nu) = \mathfrak{p}$ is called the defining ideal of the monomial curve defined by the parametrization $\nu(x_i) = t^{a_i}$, $1 \leq i \leq e$.

Let $e \geq 4$. Let us consider the string of positive integers in arithmetic progression: $a < a + d < a + 2d < \ldots < a + (n-1)d < b < b + d < \ldots < b + (m-1)d$, where $m, n \in \mathbb{N}$, $m+n = e$ and $\gcd(a, d) = 1$. Note that $a < a+d < a+2d < \ldots < a+(n-1)d$ and $b < b+d < \ldots < b+(m-1)d$ are both arithmetic sequences with the same common difference d. We further assume that this sequence minimally generates the numerical semigroup $\Gamma = \langle a, a+d, a+2d, \ldots, a+(n-1)d, b, b+d, \ldots, b+(m-1)d\rangle$. Then, Γ is called the *numerical semigroup generated by concatenation of two arithmetic sequences with the same common difference* d.

3. Main Results

In this section we give the explicit structure of the tangent cone of different classes of Numerical semigroups generated by concatenation using the Theorem 2.3 in [3]. Further we study their Hilbert series as well.

3.1. **Unbounded Class of Numerical Semigroups.** Let $e \geq 4$, $p \geq 0$ and $n = (e-1) + k(e-3)$, where $k \geq 2$. Let $\mathfrak{S}_{(n,e,p)} = \langle m_0, \ldots, m_{e-1}\rangle$, where $m_i := n^2 + (e-2)n + p + i$, for $0 \leq i \leq e-3$ and $m_{e-2} := n^2+(e-1)n+p+(e-3)$, $m_{e-1} := n^2+(e-1)n+p+(e-2)$.

If $p = e-4$, then we denote $\mathfrak{S}_{(n,e,e-4)}$ by $\mathfrak{S}_{(n,e)}$.

Theorem 3.1. *Let $I = \langle t^{n^2+2n}\rangle$. The tangent cone $G_{\mathfrak{m}}(\mathfrak{S}_{(n,4)})$ of $\mathfrak{S}_{(n,4)}$ is a free $F(I)$-module. Moreover*

$$G_{\mathfrak{m}}(\mathfrak{S}_{(n,4)}) = \bigoplus_{k=1}^{n} (F(I)(-k))^{2k+1}.$$

and $G_{\mathfrak{m}}(\mathfrak{S}_{(n,4)})$ is Cohen-Macaulay.

Corollary 3.2. *Let $I = \langle t^{n^2+3n}\rangle$. The tangent cone $G_{\mathfrak{m}}(\mathfrak{S}_{(n,5)})$ of $\mathfrak{S}_{(n,5)}$ is a free $F(I)$-module. Moreover*

$$G_{\mathfrak{m}}(\mathfrak{S}_{(n,5)}) = \bigoplus (F(I)(-1))^4 \bigoplus (F(I)(-2))^{11} \bigoplus_{m=1}^{n/4-1/2} (F(I)(-2m-1))^{6m+5}$$

$$\bigoplus_{m=n/4}^{n/2-1}(F(I)(-2m-1))^{2n-2m+7}\bigoplus_{m=2}^{n/4}(F(I)(-2m))^{6m+2}\bigoplus_{m=n/4+1/2}^{n/2}(F(I)(-2m))^{2n-2m+8}.$$

and the Tangent cone $G_{\mathfrak{m}}(\mathfrak{S}_{(n,5)})$ is Cohen-Macaulay.

3.2. **Symmetric Class of Numerical Semigroups.** Let $e \geq 4$ be an integer, q a positive integer and $m = e + 2q + 1$. Let d be a positive integer that satisfies $gcd(m, d) = 1$. Let us define $\mathcal{S} = \{m, m+d, (q+1)m+(q+2)d, (q+1)m+(q+3)d, \ldots, (q+1)m+(q+e-1)d\}$ and $\Gamma_{(e,q,d)}(\mathcal{S})$ be the numerical semigroup generated by $\mathcal{S}$.

Corollary 3.3. *Let $I = \langle t^m \rangle$. The tangent cone $G_{\mathfrak{m}}(\Gamma_{(e,q,d)}(\mathcal{S}))$ of $\Gamma_{(e,q,d)}(\mathcal{S})$ is a free $F(I)$-module. Moreover*

$$G_{\mathfrak{m}}(\Gamma_{(e,q,d)}(\mathcal{S})) = \bigoplus F(I) \bigoplus_{i=1}^{q+1} F(I)(-i) \bigoplus_{i=1}^{q+2}\left(F(I)(-(e+q+i-2)) \oplus \frac{F(I)}{(t^m)^{e-2}F(I)}(-i)\right)$$
$$\bigoplus_{i=1}^{e-3}\left(F(I)(-(q+i+1)) \oplus \frac{F(I)}{(t^m)^{i}F(I)}(-1)\right)$$

and $G_{\mathfrak{m}}(\Gamma_{(e,q,d)}(\mathcal{S}))$ is not Cohen-Macaulay.

3.3. **Almost Maximal Class of Numerical Semigroups.** Let $e \geq 4$ be an integer; $a = e+1$, $b > a + (e-3)d$, $\gcd(a, d) = 1$ and $d \nmid (b - a)$. Let $M = \{a, a + d, a + 2d, \ldots, a + (e-3)d, b, b + d\}$ and we assume that the set forms a minimal generating set for the numerical semigroup $\Gamma_e(M)$, generated by the set M.

Let us write $d \equiv i (\operatorname{mod} a)$.

Corollary 3.4. *Let $I = \langle t^a \rangle$. The tangent cone $G_{\mathfrak{m}}(\Gamma_e(M))$ of $\Gamma_e(M)$ is a free $F(I)$-module. Moreover*

(1) $G_{\mathfrak{m}}(\Gamma_4(M)) = \bigoplus(F(I)(-1))^3 \bigoplus(F(I)(-2))^1$.

(2) For $e \geq 5$, $G_{\mathfrak{m}}(\Gamma_e(M)) = \bigoplus(F(I)(-1))^{e-1} \bigoplus(F(I)(-2))^1$.

and $G_{\mathfrak{m}}(\Gamma_4(M))$ and $G_{\mathfrak{m}}(\Gamma_e(M))$, $e \geq 5$ are Cohen-Macaulay.

3.4. **Tangent cone of Bresinsky's Curves.** Let $h \geq 2$ be an even integer. Let $m_0 = 2h(2h-1)$, $m_1 = (2h+1)(2h-1)$, $m_2 = 2h(2h+1)$, $m_3 = 2h(2h+1) + (2h-1)$. Bresinsky see [5] defined the curve $\Gamma_h = \langle m_0, m_1, m_2, m_3 \rangle$.

Corollary 3.5. *Let $I = \langle t^{n_1} \rangle$. The tangent cone $G_{\mathfrak{m}}(\Gamma_h)$ of Γ_h is a free $F(I)$-module. Moreover*

$$G_{\mathfrak{m}}(\Gamma_h) = \bigoplus_{k=1}^{2h-2}(F(I)(-k))^{2k+1} \bigoplus(F(I)(2h-1))^{2h-1}.$$

and $G_{\mathfrak{m}}(\Gamma_h)$ is Cohen-Macaulay.

3.5. **Tangent cone of Arslan curve.** Let $m \geq 2$ and $n_1 = m(m+1), n_2 = m(m+1)+1, n_3 = (m+1)^2, n_4 = (m+1)^2+1$. Arslan in [4] defined the following curves $\Gamma_m = \langle n_1, n_2, n_3, n_4 \rangle$.

Corollary 3.6. *Let $I = \langle t^{n_1} \rangle$. The tangent cone $G_{\mathfrak{m}}(\Gamma_m)$ of Γ_m is a free $F(I)$-module. Moreover*

$$G_{\mathfrak{m}}(\Gamma_m) = \bigoplus_{k=1}^{m-1} (F(I)(-k))^{2m-1} \bigoplus (F(I)(-m))^m.$$

and $G_{\mathfrak{m}}(\Gamma_m)$ is Cohen-Macaulay.

Remark 3.7. Cohen-Macaulayness of the tangent cone of Bresinsky curves and Arslan curves has been already studid in [6] and [4] respectively. But here we studid Apery table and giving an explicit structure of the tangent cone of Arslan curves.

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