THE SET OF NORMALIZED IDEALS OF A NUMERICAL SEMIGROUP

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Let S be a numerical semigroup (a submonoid of the set of nonnegative integers under addition such that $\max(\mathbb{Z} \setminus S)$ exists). A set of integers I is said to be a fractional ideal of S if $I + S \subseteq I$ and I has a minimum (equivalently, $a + I \subseteq S$ for some integer a). If I and J are ideals of S, we write $I \sim J$ if there exists an integer z such that $I = z + J$. The ideal class monoid of S is defined as the set of fractional ideals of S modulo this relation. It can be easily proven that this monoid is isomorphic to the monoid of fractional ideals of S having minimum equal to zero.

The ideal class monoid of a numerical semigroup is a finite commutative monoid, which is reduced but not unit-cancellative, and it can be endowed with two preorders (inclusion and the preorder induced by addition). We describe the notable elements of the resulting posets with respect to these preorders, showing that some recover classical invariants of the numerical semigroup. We also study what are the irreducible, prime elements, quarks and atoms with respect to addition. Idempotent quarks correspond with unitary extensions of the numerical semigroup. As a consequence, we prove that a numerical semigroup is irreducible if and only if its ideal class monoid has at most two quarks.

We show that the ideal class monoid of a numerical semigroup fully characterizes the semigroup: if two numerical semigroups share isomorphic ideal class monoids, then they must be equal. To prove this, we will first show that the Hasse diagram of the gaps of a numerical semigroup, with respect to the order induced by the semigroup, completely determines the semigroup.

Given a numerical semigroup S, a subset I of S is an (integral) ideal of S if $I + S = I$. The set of ideals of S, denoted $\mathfrak{I}(S)$, is a monoid under the operation of set-wise addition. It is natural to ask if given S and T two numerical semigroup, the existence of an isomorphism between $\mathfrak{I}(S)$ and $\mathfrak{I}(T)$ forces S to be equal to T. We show that this is the case, and prove that this result can be extended to a wider class of monoids. As an application, we give an alternate proof of the well known fact that if the power monoids of S and T are isomorphic, then S and T must be equal.

This talk is an excerpt of some recent works in collaboration with L. Casabella, M. D'Anna and S. Tringali.

References

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