NEARLY GORENSTEIN LOCAL RINGS DEFINED BY MAXIMAL MINORS OF A $2 \times n$ MATRIX

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This is a joint work with Shinya Kumashiro and Taiga Nakashima [5].

In this talk, we will focus on two concepts that generalize the Gorenstein property for local rings: *ALMOST Gorenstein* property and *NEARLY Gorenstein* property. In what follows, R denotes a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$ possessing the canonical module K_R.

The notion of almost Gorenstein rings was introduced by Valentina Barucci and Ralf Fröberg [1] in 1997 for 1-dimensional analytically unramified local rings based on the deep observations of almost symmetry in numerical semigroups. Inspired by their work, Shiro Goto, Tran Thi Phuong, and I [2] defined almost Gorenstein property for arbitrary 1-dimensional Cohen-Macaulay local rings. Furthermore, Goto, Ryo Takahashi, and Naoki Taniguchi [3] extended the almost Gorenstein property for higher-dimensional Cohen-Macaulay local/graded rings. Roughly speaking, the notion of almost Gorenstein rings is defined based on the philosophy that the base ring R is embedded in the canonical module K_R, and the cokernel is controllable, which should be close to Gorenstein rings. Notice that the Gorenstein property on R is characterized by the isomorphism $R \cong \mathrm{K}_R$ as R-modules.

On the other hand, nearly Gorenstein was defined by Jürgen Herzog, Takayuki Hibi, and Dumitru I. Stamate [4] in 2019. They focused on the trace ideal

$$\mathrm{tr}_R(\mathrm{K}_R) = \sum_{f \in \mathrm{Hom}_R(\mathrm{K}_R, R)} \mathrm{Im}\, f$$

of the canonical module K_R, noting that there is an equivalence between R being Gorenstein and the equality $R = \mathrm{tr}_R(\mathrm{K}_R)$. The notion of nearly Gorenstein rings is defined by the inclusion $\mathrm{tr}_R(\mathrm{K}_R) \supseteq \mathfrak{m}$, which means that $\mathrm{tr}_R(\mathrm{K}_R)$ does not necessarily coincide with R but is still large.

Let me state the definitions precisely.

Definition 1. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing the canonical module K_R. We say that R is a *almost Gorenstein local ring*, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

such that $C = (0)$ or $\mu_R(C) = \mathrm{e}_{\mathfrak{m}}(C)$, where $\mu_R(C)$ denotes the number of a minimal system of generators of C and $\mathrm{e}_{\mathfrak{m}}(C)$ denotes the multiplicity of C with respect to $\mathfrak{m}$.

Notice that, when $\dim R = 1$, the equality $\mu_R(C) = \mathrm{e}_{\mathfrak{m}}(C)$ holds if and only if C is an $R/\mathfrak{m}$-vector space. This means that this definition of almost Gorenstein rings coincides with the definition of that in the sense of V. Barucci and R. Fröberg [1], when R is a one-dimensional analytically unramified local ring.

Definition 2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing the canonical module K_R. We say that R is a *nearly Gorenstein local ring*, if $\mathrm{tr}_R(\mathrm{K}_R) \supseteq \mathfrak{m}$.

It is known by [4] that, every one-dimensional almost Gorenstein ring is nearly Gorenstein. On the other hand, if R has minimal multiplicity, then the nearly Gorenstein property implies the almost Gorenstein property.

Let us discuss the question of how different these two properties are, especially in one dimension, more precisely, when R is a numerical semigroup ring.

In what follows, let k be a field, $n \geq 3$ a positive integer, and $S = k[[X_1, X_2, \dots, X_n]]$ the formal power series ring over k with n variables. We take positive integers $a_1, a_2, \dots, a_n \geq 1$ with $\gcd(a_1, a_2, \dots, a_n) = 1$ and put $H = \langle a_1, a_2, \dots, a_n \rangle$ the numerical semigroup minimally generated by $a_1, a_2, \dots, a_n$. Then, the numerical semigroup ring $k[[H]]$ of H is defined as

$$k[[H]] = k[[t^h \mid h \in H]] = k[[t^{a_1}, t^{a_2}, \dots, t^{a_n}]] \subseteq k[[t]],$$

where $k[[t]]$ denotes the formal power series ring over k with a variable t. Moreover, we consider the ring homomorphism $\varphi_H : S \to k[[H]]$ defined by $\varphi_H(X_i) = t^{a_i}$ and put $I_H = \mathrm{Ker}\, \varphi_H \subseteq S$. The ideal I_H is called the defining ideal of $k[[H]]$. Thereafter, we further assume that

$$I_H = \mathrm{I}_2 \begin{pmatrix} X_2^{m_2} & X_3^{m_3} & \cdots & X_n^{m_n} & X_1^{m_1} \\ X_1^{\ell_1} & X_2^{\ell_2} & \cdots & X_{n-1}^{\ell_{n-1}} & X_n^{\ell_n} \end{pmatrix}$$

for some positive integers $m_1, m_2, \ldots, m_n, \ell_1, \ell_2, \ldots, \ell_n \geq 1$, where $\mathrm{I}_2(\mathsf{X})$ denotes the ideal generated by all 2×2 minors of a matrix X.

Under this setting, the main result of this talk is the following.

Theorem 3. *$R = k[[H]]$ is a nearly Gorenstein ring if and only if, after suitable permutation of the minimal system $a_1, a_2, \ldots, a_n$ of generators H, one of the following is satisfied.*

(1) $m_1 = m_2 = \cdots = m_n = 1$.
(2) $m_2 = m_3 = \cdots = m_n = \ell_1 = \ell_2 = \cdots = \ell_{n-2} = 1$.

Under our assumption, notice that the condition (1) is equivalent to R being almost Gorenstein. Recalling that in the one-dimensional case, every almost Gorenstein ring is nearly Gorenstein, the difference between the two conditions is summed up in the condition (2).

In the talk, let me give a few remarks on the proof of this theorem. If time permits, let me discuss a higher-dimensional analog of such kinds of rings.

References

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