PROJECTIVE MONOMIAL CURVES AND THEIR AFFINE PROJECTIONS

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ABSTRACT. In this work, we explore when the Betti numbers of the coordinate rings of a projective monomial curve and one of its affine projections are identical. Given an infinite field k and a sequence of relatively prime integers $a_0 = 0 < a_1 < \cdots < a_n = d$, we consider the projective monomial curve $\mathcal{C} \subset \mathbb{P}_k^n$ of degree d parametrically defined by $x_i = u^{a_i} v^{d-a_i}$ for all $i \in \{0, \ldots, n\}$ and its coordinate ring $k[\mathcal{S}]$. The curve $\mathcal{C}_1 \subset \mathbb{A}_k^n$ with parametric equations $x_i = t^{a_i}$ for $i \in \{1, \ldots, n\}$ is an affine projection of $\mathcal{C}$ and we denote by $k[\mathcal{S}_1]$ its coordinate ring. The main contribution of this paper is the introduction of a novel (Gröbner-free) combinatorial criterion that provides a sufficient condition for the equality of the Betti numbers of $k[\mathcal{C}]$ and $k[\mathcal{C}_1]$. Leveraging this criterion, we identify infinite families of projective curves satisfying this property.

INTRODUCTION

Let k be an infinite field, and $k[\mathbf{x}] := k[x_1, \ldots, x_n]$ and $k[\mathbf{t}] := k[t_1, \ldots, t_m]$ be two polynomial rings over k. Given $\mathcal{B} = \{b_1, \ldots, b_n\} \subset \mathbb{N}^m$, a set of nonzero vectors, each element $b_i = (b_{i1}, \ldots, b_{im}) \in \mathbb{N}^m$ corresponds to the monomial $\mathbf{t}^{b_i} := t_1^{b_{i1}} \cdots t_m^{b_{im}} \in k[\mathbf{t}]$. The affine toric variety $X_\mathcal{B} \subset \mathbb{A}_k^n$ determined by $\mathcal{B}$ is the Zariski closure of the set given parametrically by $x_i = u_1^{b_{i1}} \cdots u_m^{b_{im}}$ for all $i = 1, \ldots, n$. Consider

$$\mathcal{S}_\mathcal{B} := \langle b_1, \ldots, b_n \rangle = \{\alpha_1 b_1 + \cdots + \alpha_n b_n \,|\, \alpha_1, \ldots, \alpha_n \in \mathbb{N}\} \subset \mathbb{N}^m ,$$

the affine monoid spanned by $\mathcal{B}$. The toric ideal determined by $\mathcal{B}$ is the kernel $I_\mathcal{B}$ of the k-algebra homomorphism $\varphi_\mathcal{B} : k[\mathbf{x}] \longrightarrow k[\mathbf{t}]$ induced by $x_i \mapsto \mathbf{t}^{b_i}$. Since k is infinite, one has that $I_\mathcal{B}$ is the vanishing ideal of $X_\mathcal{B}$ and, hence, the coordinate ring of $X_\mathcal{B}$ is (isomorphic to) the semigroup algebra $k[\mathcal{S}_\mathcal{B}] := \operatorname{Im}(\varphi_\mathcal{B}) \simeq k[\mathbf{x}]/I_\mathcal{B}$. The ideal $I_\mathcal{B}$ is an $\mathcal{S}_\mathcal{B}$-homogeneous binomial ideal, i.e., if one sets the $\mathcal{S}_\mathcal{B}$-degree of a monomial $\mathbf{x}^\alpha \in k[\mathbf{x}]$ as $\deg_{\mathcal{S}_\mathcal{B}}(\mathbf{x}^\alpha) := \alpha_1 b_1 + \cdots + \alpha_n b_n \in \mathcal{S}_\mathcal{B}$, then $I_\mathcal{B}$ is generated by $\mathcal{S}_\mathcal{B}$-homogeneous binomials. One can thus consider a minimal $\mathcal{S}_\mathcal{B}$-graded free resolution of $k[\mathcal{S}_\mathcal{B}]$ as $\mathcal{S}_\mathcal{B}$-graded $k[\mathbf{x}]$-module,

$$\mathcal{F} : 0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_0 \longrightarrow k[\mathcal{S}_\mathcal{B}] \longrightarrow 0 .$$

The projective dimension of $k[\mathcal{S}_\mathcal{B}]$ is $\operatorname{pd}(k[\mathcal{S}_\mathcal{B}]) = \max\{i \,|\, F_i \neq 0\}$. The i-th Betti number of $k[\mathcal{S}_\mathcal{B}]$ is the rank of the free module F_i, i.e., $\beta_i(k[\mathcal{S}_\mathcal{B}]) = \operatorname{rank}(F_i)$; and the Betti sequence of $k[\mathcal{S}_\mathcal{B}]$ is $(\beta_i(k[\mathcal{S}_\mathcal{B}])\,;\, 0 \leq i \leq \operatorname{pd}(k[\mathcal{S}_\mathcal{B}]))$. When the Krull dimension of $k[\mathcal{S}_\mathcal{B}]$ coincides with its depth as $k[\mathbf{x}]$-module, the ring $k[\mathcal{S}_\mathcal{B}]$ is said to be Cohen-Macaulay. By the Auslander-Buchsbaum formula, this is equivalent to $\operatorname{pd}(k[\mathcal{S}_\mathcal{B}]) = n - \dim(k[\mathcal{S}_\mathcal{B}])$. When $k[\mathcal{S}_\mathcal{B}]$ is Cohen-Macaulay, its (Cohen-Macaulay) type is the rank of the last nonzero module in the resolution, i.e., $\operatorname{type}(k[\mathcal{S}_\mathcal{B}]) := \beta_p(k[\mathcal{S}_\mathcal{B}])$ where $p = \operatorname{pd}(k[\mathcal{S}_\mathcal{B}])$.

Now consider $d \in \mathbb{Z}^+$ and $a_0 := 0 < a_1 < \cdots < a_n = d$ a sequence of relatively prime integers. Denote by $\mathcal{C}$ the projective monomial curve $\mathcal{C} \subset \mathbb{P}_k^n$ of degree d parametrically defined by

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$x_i = u^{a_i} v^{d-a_i}$ for all $i \in \{0, \ldots, n\}$, i.e., $\mathcal{C}$ is the Zariski closure of

$$\{(u^{a_0} v^{d-a_0} : \cdots : u^{a_i} v^{d-a_i} : \cdots : u^{a_n} v^{d-a_n}) \in \mathbb{P}_k^n \,|\, (u : v) \in \mathbb{P}_k^1\}.$$

Taking $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^2$ with $\mathbf{a}_i = (a_i, d-a_i)$ for all $i = 0, \ldots, n$, one has that $I_{\mathcal{A}}$ is the vanishing ideal of $\mathcal{C}$, and the coordinate ring of $\mathcal{C}$ is the two-dimensional ring $k[\mathcal{S}] \simeq k[x_0, \ldots, x_n]/I_{\mathcal{A}}$, where $\mathcal{S} = \mathcal{S}_{\mathcal{A}}$ denotes the monoid spanned by $\mathcal{A}$. The projective monomial curve $\mathcal{C}$ is said to be arithmetically Cohen-Macaulay if the ring $k[\mathcal{S}]$ is Cohen-Macaulay.

The projective curve $\mathcal{C}$ has two affine projections, $\mathcal{C}_1 = \{(u^{a_1}, \ldots, u^{a_n}) \in \mathbb{A}_k^n \,|\, u \in k\}$ and $\mathcal{C}_2 = \{(v^{d-a_0}, v^{d-a_1}, \ldots, v^{d-a_{n-1}}) \in \mathbb{A}_k^n \,|\, v \in k\}$, associated to the sequences $a_1 < \cdots < a_n$ and $d - a_{n-1} < \cdots < d - a_1 < d - a_0$, respectively. The second sequence is sometimes called the dual of the first one. Denote by $\mathcal{S}_1 := \mathcal{S}_{\mathcal{A}_1}$ the numerical semigroup generated by $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$. The vanishing ideal of $\mathcal{C}_1$ is $I_{\mathcal{A}_1} \subset k[x_1, \ldots, x_n]$, and hence, its coordinate ring is the one-dimensional ring $k[\mathcal{S}_1] \simeq k[x_1, \ldots, x_n]/I_{\mathcal{A}_1}$. Moreover, $I_{\mathcal{A}}$ is the homogenization of $I_{\mathcal{A}_1}$ with respect to the variable x_0. Similarly, denoting by $\mathcal{S}_2 := \mathcal{S}_{\mathcal{A}_2}$ the numerical semigroup generated by $\mathcal{A}_2 := \{d - a_0, d - a_1, \ldots, d - a_{n-1}\}$, the vanishing ideal of $\mathcal{C}_2$ is $I_{\mathcal{A}_2} \subset k[x_0, \ldots, x_{n-1}]$, its coordinate ring is $k[\mathcal{S}_2] \simeq k[x_0, \ldots, x_{n-1}]/I_{\mathcal{A}_2}$, and $I_{\mathcal{A}}$ is the homogenization of $I_{\mathcal{A}_2}$ with respect to x_n.

One has that $\beta_i(k[\mathcal{S}]) \geq \beta_i(k[\mathcal{S}_1])$ for all i, and the goal of this work is to understand when the Betti sequences of $k[\mathcal{S}]$ and $k[\mathcal{S}_1]$ coincide. A necessary condition is that $k[\mathcal{S}]$ is Cohen-Macaulay. Indeed, affine monomial curves are always arithmetically Cohen-Macaulay while projective ones may be arithmetically Cohen-Macaulay or not. Thus, $\mathrm{pd}(k[\mathcal{S}]) = \mathrm{pd}(k[\mathcal{S}_1])$ if and only if $\mathcal{C}$ is arithmetically Cohen-Macaulay. In Theorem 1.2, which is the main result of this work, we provide a combinatorial sufficient condition for having equality between the Betti sequences of $k[\mathcal{S}]$ and $k[\mathcal{S}_1]$ by means of the poset structures induced by $\mathcal{S}$ and $\mathcal{S}_1$ on the Apery sets of both $\mathcal{S}$ and $\mathcal{S}_1$. In Propositions 2.1 and 2.2, we use our main result to provide explicit families of curves where $\beta_i(k[\mathcal{S}]) = \beta_i(k[\mathcal{S}_1])$ for all i.

The motivation of this work comes from [4], where the authors obtain a sufficient condition in terms of Gröbner bases to ensure the equality of the Betti sequences.

1. Apery sets and Betti numbers

Let $d \in \mathbb{Z}^+$ and $a_0 := 0 < a_1 < \cdots < a_n = d$ be a sequence of relatively prime integers. For each $i = 0, \ldots, n$, set $\mathbf{a}_i := (a_i, d - a_i) \in \mathbb{N}^2$, and consider the three sets $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$, $\mathcal{A}_2 = \{d, d - a_1, \ldots, d - a_{n-1}\}$ and $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^2$. We denote by $\mathcal{C} \subset \mathbb{P}_k^n$ the projective monomial curve defined by $\mathcal{A}$ as defined in the introduction, and by $\mathcal{C}_1$ and $\mathcal{C}_2$ its affine projections. Consider $\mathcal{S}_1$ and $\mathcal{S}_2$ the numerical semigroups generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, and $\mathcal{S}$ the monoid spanned by $\mathcal{A}$ that we call the homogenization of $\mathcal{S}_1$ (with respect to d).

Definition 1.1. For $i = 1, 2$, the Apery set of $\mathcal{S}_i$ with respect to d is $\mathrm{Ap}_i := \{y \in \mathcal{S}_i \,|\, y - d \notin \mathcal{S}_i\}$. One can also define the Apery set of $\mathcal{S}$ as $\mathrm{AP}_{\mathcal{S}} := \{\mathbf{y} \in \mathcal{S} \,|\, \mathbf{y} - \mathbf{a}_0 \notin \mathcal{S}, \mathbf{y} - \mathbf{a}_n \notin \mathcal{S}\}$.

Note that $\mathrm{AP}_{\mathcal{S}}$ has at least d elements by [3, Lem. 2.5]. Moreover, $|\mathrm{AP}_{\mathcal{S}}| = d$ if and only if $\mathcal{C}$ is arithmetically Cohen-Macaulay.

In order to compare $\beta_i(k[\mathcal{S}])$ and $\beta_i(k[\mathcal{S}_1])$ for all i, we will relate in Theorem 1.2 the Apery sets Ap_1 and $\mathrm{AP}_{\mathcal{S}}$ with the natural poset structure that both have and that we now define. For $i = 1, 2$, $(\mathrm{Ap}_i, \leq_i)$ is a poset, where $\leq_i$ is given by $y \leq_i z \Longleftrightarrow z - y \in \mathcal{S}_i$. Similarly, $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ is a poset for $\leq_{\mathcal{S}}$ defined by $\mathbf{y} \leq_{\mathcal{S}} \mathbf{z} \Longleftrightarrow \mathbf{z} - \mathbf{y} \in \mathcal{S}$.

The main result in this section is Theorem 1.2 where we give a sufficient condition in terms of the poset structures of the Apery sets Ap_1 and $\mathrm{AP}_{\mathcal{S}}$ for the Betti sequences of $k[\mathcal{S}_1]$ and $k[\mathcal{S}]$ to coincide.

Theorem 1.2. *If* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1)$*, then* $\beta_i(k[\mathcal{S}]) = \beta_i(k[\mathcal{S}_1])$ *for all* i*.*

In fact, the condition $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1)$ can be checked in terms of the poset Ap_1 and the minimal generators of $\mathcal{S}_1$ when $k[\mathcal{S}]$ is Cohen-Macaulay, as shown in Proposition 1.4. Before stating that result, let us first recall some useful notions about posets.

Definition 1.3. Let $(\Sigma, \leq)$ be a poset.

(a) For $y, z \in \Sigma$, we say that z *covers* y, and denote it by $y \prec z$, if $y < z$ and there is no $w \in \Sigma$ such that $y < w < z$.
(b) We say that Σ is *graded* if there exists a function $\rho : \Sigma \to \mathbb{N}$, called rank function in Σ, such that:
  - If $y, z \in \Sigma$ and $y < z$, then $\rho(y) < \rho(z)$.
  - If $y, z \in \Sigma$ and $y \prec z$, then $\rho(z) = \rho(y) + 1$.

Proposition 1.4. *The following two claims are equivalent:*

(a) The posets $(\mathrm{Ap}_1, \leq_1)$ *and* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ *are isomorphic;*
(b) $k[\mathcal{S}]$ *is Cohen-Macaulay,* $(\mathrm{Ap}_1, \leq_1)$ *is graded, and* $\{a_1, \ldots, a_{n-1}\}$ *is contained in the minimal system of generators of* $\mathcal{S}_1$*.*

2. Examples of application

In Propositions 2.1 and 2.2, we provide some sequences $a_1 < \cdots < a_n$ for which the condition in Theorem 1.2 is satisfied. Let us start with arithmetic sequences, i.e., sequences $a_1 < \cdots < a_n$ such that $a_i = a_1 + (i-1)e$ for some positive integer e. For this family, we refine [4, Cor. 4.2] that considers $a_1 > n - 1$.

Proposition 2.1. *Let* $a_1 < \ldots < a_n$ *be an arithmetic sequence of relatively prime integers. Then,* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1)$ *if and only if* $a_1 > n - 2$*. Therefore, if* $a_1 > n - 2$*, the Betti sequences of* $k[\mathcal{S}_1]$ *and* $k[\mathcal{S}]$ *coincide.*

Example 1. For the sequence $5 < 6 < 7 < 8 < 9 < 10$, one has that $a_1 = 5 > 4 = n - 2$. Therefore, the Apery sets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are isomorphic. Hence, by Theorem 1.2, the Betti sequences of $k[\mathcal{S}_1]$ and $k[\mathcal{S}]$ coincide. One can check that both are $(1, 11, 30, 35, 19, 4)$. The posets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ in this example are shown in Figure 1.

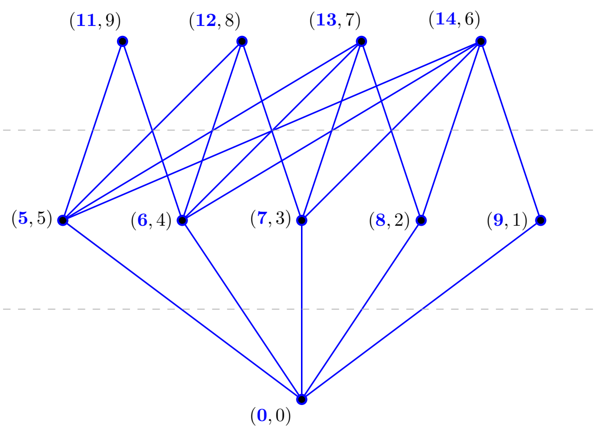

FIGURE 1. The posets $(\mathrm{Ap}_1, \leq_1)$ (in blue) and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ (in black) for $\mathcal{S}_1 = \langle 5, 6, 7, 8, 9, 10 \rangle$.

In [1, Sect. 6], the authors studied the canonical projections of the projective monomial curve $\mathcal{C}$ defined by an arithmetic sequence $a_1 < \cdots < a_n$ of relatively prime integers, i.e., the curve $\pi_r(\mathcal{C})$

obtained as the Zariski closure of the image of $\mathcal{C}$ under the r-th canonical projection $\pi_r : \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^{n-1}$, $(p_0 : \cdots : p_n) \mapsto (p_0 : \cdots : p_{r-1} : p_{r+1} : \cdots : p_n)$. We know that $\pi_r(\mathcal{C})$ is the projective monomial curve associated to the sequence $a_1 < \cdots < a_{r-1} < a_{r+1} < \cdots < a_n$.

In Proposition 2.2, for any $r \in \{2, \ldots, n-1\}$, we consider $\mathcal{A}_1 = \{a_1, \ldots, a_n\} \setminus \{a_r\}$, the numerical semigroup $\mathcal{S}_1 = \mathcal{S}_{\mathcal{A}_1}$, and its homogenization $\mathcal{S}$, and we characterize when the posets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are isomorphic.

Proposition 2.2. *Let $a_1 < \ldots < a_n$ be an arithmetic sequence of relatively prime integers with $n \geq 4$, and $r \in \{2, \ldots, n-1\}$. Consider $\mathcal{A}_1 = \{a_1, \ldots, a_n\} \setminus \{a_r\}$, the numerical semigroup $\mathcal{S}_1$ generated by $\mathcal{A}_1$, and its homogenization $\mathcal{S}$. Then the posets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are isomorphic if and only if one of the following conditions holds:*

(a) $r = 2$, $a_1 \geq n-1$ *and* $a_1 \neq n$;
(b) $3 \leq r \leq n-2$, $a_1 \geq n$ *and* $r \leq a_1 - n + 1$;
(c) $r = n-1$ *and* $a_1 \geq n-2$.

Consequently, if one of the previous conditions holds, then $\beta_i(k[\mathcal{S}_1]) = \beta_i(k[\mathcal{S}])$, for all i.

Example 2. For the sequence $9 < 10 < 11 < 12 < 13$, the Betti sequences of $k[\mathcal{S}_1]$ and $k[\mathcal{S}]$ coincide by Proposition 2.1. Indeed, it is $(1, 10, 20, 15, 4)$ for both curves. The parameters of this arithmetic sequence are $a_1 = 9$, $e = 1$ and $n = 5$. Hence, for $r = 2, 3, 4$, if $\mathcal{A}_1 = \{9, 10, 11, 12, 13\} \setminus \{a_r\}$, the Betti sequences of $k[\mathcal{S}_1]$ and $k[\mathcal{S}]$ coincide by Proposition 2.2. One can check that the Betti sequence is $(1, 5, 6, 2)$ when $r = 2$ or $r = 4$, and it is $(1, 8, 12, 5)$ when $r = 3$.

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