PROJECTIVE CLOSURE OF SEMIGROUP ALGEBRAS

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ABSTRACT. This paper investigates the projective closure of simplicial affine semigroups in $\mathbb{N}^d$, $d \geq 2$. We present a characterization of the Cohen-Macaulay property for the projective closure of these semigroups using Gröbner bases. Additionally, we establish a criterion, based on Gröbner bases, for determining the Buchsbaum property of non-Cohen-Macaulay projective closures of numerical semigroup rings. Lastly, we introduce the concept of k-lifting for simplicial affine semigroups in $\mathbb{N}^d$, and investigate its relationship with the original simplicial affine semigroup.

1. INTRODUCTION

Let Γ be an affine semigroup, fully embedded in $\mathbb{N}^d$. Let Γ be minimally generated by the set $\{\mathbf{m}_1, \ldots, \mathbf{m}_{d+r}\}$. The semigroup algebra $\mathbb{K}[\Gamma]$ over a field $\mathbb{K}$ is generated by the monomials $\mathbf{x}^{\mathbf{m}}$, where $\mathbf{m} \in \Gamma$, with maximal ideal $\mathfrak{m} = (\mathbf{x}^{\mathbf{m_1}}, \ldots, \mathbf{x}^{\mathbf{m_{d+r}}})$. Let $I(\Gamma)$ denote the defining ideal of $\mathbb{K}[\Gamma]$, which is the kernel of the $\mathbb{K}$=algebra homomorphism $\phi : A = \mathbb{K}[z_1, \ldots, z_{d+r}] \to \mathbb{K}[\mathbf{s}]$, such that $\phi(z_i) = \mathbf{s}^{\mathbf{m_i}}$, $i = 1, \ldots, d+r$ and $\mathbf{s} = s_1 \ldots s_d$. Let us write $\mathbb{K}[\Gamma] \cong A/I(\Gamma)$. The defining ideal $I(\Gamma)$ is a binomial prime ideal.

Consider a partial relation order on $\mathbb{N}^d$: $\mathbf{a} = (a_1, \ldots, a_d) \leq \mathbf{b} = (b_1, \ldots, b_d)$ if and only if $a_i \leq b_i, \forall i$. For indeterminate $\mathbf{s} = s_1 \ldots s_d$, $\mathbf{t} = t_1 \ldots t_d$, consider a map $\phi^h : \mathbb{K}[z_0, z_1, \ldots, z_{d+r}] \to \mathbb{K}[\mathbf{s}, \mathbf{t}]$ defined by $\phi^h(z_i) = \mathbf{s}^{\mathbf{m}_{d+r} - \mathbf{m}_i}\mathbf{t}^{m_i}$ for $1 \leq i \leq d+r$ and $\phi^h(z_0) = \mathbf{s}^{\mathbf{m_{d+r}}}$. The image of the map ϕ^h is the subalgebra $\mathbb{K}[\mathcal{A}]$ of $\mathbb{K}[\mathbf{s}, \mathbf{t}]$ generated by the monomials whose exponents are the generators of the affine semigroup

$$\Gamma^h = \langle \{(\mathbf{m_{d+r}}, \mathbf{0}), (\mathbf{m_{d+r}} - \mathbf{m_1}, \mathbf{m}_1), \ldots, (\mathbf{m_{d+r}} - \mathbf{m_{d+r-1}}, \mathbf{m}_{d+r-1}), (\mathbf{0}, \mathbf{m}_{d+r})\} \rangle.$$

We call the subalgebra $\mathbb{K}[\Gamma^h]$ the projective closure of the affine semigroup Γ. Its vanishing ideal $I(\Gamma^h)$ is given by the kernel of the homomorphism ϕ^h.

The projective closure of the affine semigroup Γ is Cohen-Macaulay if its vanishing ideal $I(\Gamma^h)$ is a Cohen-Macaulay ideal if and only if $\mathbf{s}^{\mathbf{m_{d+r}} - \mathbf{m_1}}\mathbf{t}^{\mathbf{m_1}}, \ldots, \mathbf{s}^{\mathbf{m_{d+r}} - \mathbf{m_d}}\mathbf{t}^{\mathbf{m_d}}, \mathbf{s}^{\mathbf{m_{d+r}}}$ is a regular sequence in $\mathbb{K}[\mathcal{A}]$ (see [6, Theorem 2.6]).

Affine semigroups provide a natural extension of numerical semigroups and play a crucial role in defining the projective closure of numerical semigroup rings. Gröbner basis theory has proven to be a valuable technique for studying properties related to numerical semigroups and their projective closures, including the Cohen-Macaulayness of their

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associated graded rings and the algorithm for finding the Frobenius number of numerical semigroups [1, 10]. Subsequently, Herzog and Stamate [7] established Gröbner basis criteria for determining the Cohen-Macaulay property of projective closures of affine monomial curves.

In this paper, we investigate the projective closure of simplicial affine semigroups in $\mathbb{N}^d$, where $d \geq 2$. We provide a criterion for characterizing the Cohen-Macaulayness of the projective closure of simplicial affine semigroups using the Gröbner basis of the defining ideal $I(\Gamma^h)$.

Furthermore, we explore the Buchsbaum property of the corresponding projective closures in cases where the projective closure of the numerical semigroup ring is non-Cohen-Macaulay. Kamoi [9] examined the Buchsbaum property of simplicial affine semigroups by utilizing the Gröbner basis of the corresponding defining ideal. In Section 3.2, we present a characterization of the Buchsbaum property for non-Cohen-Macaulay projective closures using the Gröbner basis of the corresponding defining ideal.

In Section 5, we study the lifting of simplicial affine semigroups in $\mathbb{N}^d$. This construction is motivated by Şahin's work [14] on k-lifting of monomial curves for a given $k \in \mathbb{N}$. It is inspired by the shifting of simplicial affine semigroups. For $d = 1$, Herzog and Stamate investigated the Cohen-Macaulayness of the associated graded ring of numerical semigroup rings. Similarly, Şahin demonstrated that if the associated graded ring of the original numerical semigroup ring is Cohen-Macaulay, then the associated graded ring of its k-lifting is also Cohen-Macaulay. We establish that the same result holds for the lifting of simplicial affine semigroups when $d \geq 2$. If Γ is a simplicial affine semigroup and Γ_k represents the corresponding k-lifting of Γ, the Betti numbers of both the rings are identical, written as $\beta_i(\mathbb{K}[\Gamma]) = \beta_i(\mathbb{K}[\Gamma_k])$, where $\beta_i(\mathbb{K}[\Gamma])$ and $\beta_i(\mathbb{K}[\Gamma_k])$ denote the i^{th} Betti numbers of $\mathbb{K}[\Gamma]$ and $\mathbb{K}[\Gamma_k]$ respectively.

2. Preliminaries

For an affine semigroup Γ, consider the natural partial ordering $\prec_\Gamma$ on $\mathbb{N}$, such that for all elements $x, y \in \mathbb{N}^r$, $x \prec_\Gamma y$ if $y - x \in \Gamma$.

Definition 2.1 ([8]). Let $\mathbf{b} \in \max_{\prec_\Gamma} \mathrm{Ap}(\Gamma, E)$; the element $\mathbf{b} - \sum_{i=1}^{r} \mathbf{a}_i$ is called a *quasi-Frobenius* element of Γ. The set of all quasi-Frobenius elements of Γ is denoted by $\mathrm{QF}(\Gamma)$ and its cardinality is said to be the *type* of Γ, denoted by $\mathrm{type}(\Gamma)$.

Remark 2.2 ([8, Proposition 3.3]). If $\mathbb{K}[\Gamma]$ is arithmetically Cohen-Macaulay, then the last Betti number of $\mathbb{K}[\Gamma]$ is called the Cohen-Macaulay type of $\mathbb{K}[\Gamma]$, written as $\mathrm{CMtype}(\mathbb{K}[\Gamma])$. Moreover, $\mathrm{type}(\Gamma) = \mathrm{CMtype}(\mathbb{K}[\Gamma])$.

Theorem 2.3. *The defining ideal $I(\Gamma^h)$ of $\mathbb{K}[\Gamma^h]$ is given by the homogenization of f with respect to variable z_0, where $f \in I(\Gamma)$, i.e., $I(\Gamma^h) = \{f^h : f \in I(\Gamma)\}$.*

3. Cohen-Macaulayness of Projective closure of simplicial affine semigroups

In this section, we study the Cohen-Macaulay characterization of the projective closure of simplicial affine semigroups in $\mathbb{N}^d$ via Gröbner basis of defining ideal of corresponding semigroups.

Theorem 3.1. *Let $E_\Gamma = \{\mathbf{m_1}, \dots, \mathbf{m_d}\}$ be a set of extremal rays of Γ. Then the set*

$$E_{\Gamma^h} = \{(\mathbf{m}_{d+r} - \mathbf{m}_i, \mathbf{m}_i), (\mathbf{m}_{d+r}, \mathbf{0}) | 1 \leq i \leq d\}$$

is set of extremal rays of Γ^h.

Let $\prec$ denotes the degree reverse lexicographic ordering on $A = \mathbb{K}[z_1, \dots, z_{d+r}]$ induced by $z_1 \prec \cdots \prec z_{d+r}$ and $\prec_0$ the induced reverse lexicographic order on $A[z_0]$, where $z_0 \prec z_1$. The following theorem is one of the main results of this paper. For any monomial ideal I, we let $G(I)$ denote the unique minimal set of monomial generators for I.

Theorem 3.2. *Let Γ be a simplicial affine semigroup in $\mathbb{N}^d, d \geq 2$ with set of extremal rays $E = \{\mathbf{m}_1, \dots, \mathbf{m}_d\}$. Then, the followings are equivalent.*
(a) $\mathbb{K}[\Gamma]$ is arithmetically Cohen-Macaulay.
(b) $\mathbb{K}[\Gamma^h]$ is arithmetically Cohen-Macaulay.
(c) $z_0, z_1, \dots, z_d$ do not divide any element of $G(\mathrm{in}_{\prec_0}(I(\Gamma^h))$.
(d) $z_1, \dots, z_d$ do not divide any element of $G(\mathrm{in}_{\prec}(I(\Gamma))$.

Theorem 3.3. *Let $\mathbb{K}[\Gamma^h]$ is arithmetically Cohen-Macaulay. The Apéry set of Γ^h with respect to E_{Γ^h} is given by* $\mathrm{Ap}(\Gamma^h, E_{\Gamma^h}) =$

$$\left\{ \sum_{i=d+1}^{d+r} \alpha_i(\mathbf{m}_{d+r} - \mathbf{m}_i, \mathbf{m}_i) : \sum_{i=d+1}^{d+r} \alpha_i \mathbf{m}_i \in \mathrm{Ap}(\Gamma, E_\Gamma) \text{ for some } (\alpha_{d+1}, \dots, \alpha_{d+r}) \in \mathbb{N}^r \right\}.$$

For an affine semigroup Γ, consider the partial natural ordering $\preceq_\Gamma$ on $\mathbb{N}^d$ where, for all elements $\mathbf{x} \preceq_\Gamma \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in \Gamma$.

Theorem 3.4. *Let $\mathbb{K}[\Gamma^h]$ be Cohen-Macaulay. Then, the Cohen-Macaulay type of $\mathbb{K}[\Gamma^h]$ is* $\mathrm{r}(\mathbb{K}[\Gamma^h]) = |\{\max_{\preceq_{\Gamma^h}}(\mathrm{Ap}(\Gamma^h, E_{\Gamma^h}))\}|$.

4. Buchsbaum Criterion for non-Cohen Macaulay Projective Closure of Numerical Semigroup rings

Theorem 4.1. *$\mathbb{K}[\Gamma^h]$ is Buchsbaum if and only if x_0, x_{s+1} do not divide the leading monomial of any element of G.*

Example 4.2. *Let $R = \mathbb{K}[u^4, u^3v, uv^3, v^4]$. Here $T = \langle (0,4), (3,1), (1,3), (0,4)\rangle$ and $T^\star$ is minimally generated by the set $\langle (0,4), (1,3), (2,2), (3,1), (4,0)\rangle$ with extremal rays $\{(0,4), (4,0)\}$. Note that $G = \{x_3^2 - x_4x_0, x_2x_3 - x_1x_0, x_1x_3 - x_2x_4, x_2^2 - x_3x_0, x_1x_2 - x_4x_0, x_1^2 - x_3x_4\}$ is the reduced Gröbner basis of $I_{T_\star}$ with respect to degree reverse lexicographic ordering $>$ induced by $x_1 > x_2 > x_3 > x_4 > x_0$. We can see that x_0, x_4 do not divide the leading monomial of any element of G, hence R is Buchsbaum ring.*

5. Lifting of affine semigroups

Let Γ be a simplicial affine semigroup minimally generated by $\{\mathbf{a}_1, \dots, \mathbf{a}_{d+r}\}$ with the set of extremal rays $\{a_1, \dots, a_d\}$. The k lifting of Γ is defined as the affine semigroup Γ_k minimally generated by $\{\mathbf{a}_1, \dots, \mathbf{a}_d, k\mathbf{a}_{d+1}, \dots, k\mathbf{a}_{d+r}\}$. Note that Γ_k is simplicial affine semigroup. Let $x_E^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$, $x_{E_c}^{\alpha'} = x_{d+1}^{\alpha_{d+1}} \dots x_{d+r}^{\alpha_{d+r}}$ where $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha' = (\alpha_{d+1}, \dots, \alpha_{d+r})$,

Theorem 5.1. *Let* $x_E^{\alpha}x_{E_c}^{\alpha'} - x_E^{\beta}x_{E_c}^{\beta'} \in I_\Gamma$ *then* $x_E^{k\alpha}x_{E_c}^{\alpha'} - x_E^{k\beta}x_{E_c}^{\beta'} \in I_{\Gamma_k}$. *Moreover,* $\mu(I_\Gamma) = \mu(I_{\Gamma_k})$.

Corollary 5.2. *If* $\mathbb{K}[\Gamma]$ *is Cohen-Macaulay (respectively Gorenstein) then* $\mathbb{K}[\Gamma_k]$ *is Cohen-Macaulay (respectively Gorenstein).*

Theorem 5.3. *If* $\mathbb{K}[\Gamma]$ *is Cohen-Macaulay. Then* $\mathrm{Ap}(\Gamma_k, E) = \{k\mathbf{b} : \mathbf{b} \in \mathrm{Ap}(\Gamma_k, E)\}$.

Definition 5.4. A semigroup Γ is said to be *of homogeneous type* if $\beta_i(\mathbb{K}[\Gamma]) = \beta_i(\mathrm{gr}_{\mathfrak{m}}(\mathbb{K}[\Gamma]))$ for all $i \geq 1$.

Theorem 5.5. *Let* $\mathrm{gr}_{\mathfrak{m}}(\mathbb{K}[\Gamma])$ *is Cohen-Macaulay and* Γ *is of homogeneous type. Then* Γ_k *is of homogeneous type.*

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