THE MULTIPLES OF A NUMERICAL SEMIGROUP

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Extended abstract

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$. A *submonoid* of $(\mathbb{N}, +)$ is a subset of $\mathbb{N}$ containing 0 and closed under addition.

If A is a non-empty subset of $\mathbb{N}$, then we write $\langle A \rangle$ for the submonoid of $(\mathbb{N}, +)$ generated by A, that is,

$$\langle A \rangle = \{u_1 a_1 + \ldots + u_n a_n \mid n \in \mathbb{N} \setminus \{0\}, \{a_1, \ldots, a_n\} \subseteq A \text{ and } \{u_1, \ldots, u_n\} \subset \mathbb{N}\}.$$

If M is a submonoid of $(\mathbb{N}, +)$ and $M = \langle A \rangle$ for some $A \subseteq \mathbb{N}$, we say that A is a *system of generators* of M. Moreover, if $M \neq \langle B \rangle$ for every $B \subsetneq A$, then we say that A is a *minimal system of generators* of M. In [10, Corollary 2.8] it is shown that every submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators which is also finite. We write $\operatorname{msg}(M)$ for the minimal system of generators of M. The cardinality of $\operatorname{msg}(M)$ is called the *embedding dimension* of M and is denoted $\operatorname{e}(M)$.

A *numerical semigroup*, S, is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus S$ is finite. The set of numerical semigroups is denoted by $\mathscr{S}$. In [10, Lemma 2.1] it is shown that a submonoid M of $(\mathbb{N}, +)$ is a numerical semigroup if and only if $\gcd(\operatorname{msg}(M)) = 1$.

If S is a numerical semigroup, then both $\operatorname{F}(S) = \max(\mathbb{Z} \setminus S)$ and $\operatorname{g}(S) = \#(\mathbb{N} \setminus S)$, where $\#$ stands for cardinality, are two important invariants of S called *Frobenius number* and *genus*, respectively.

The Frobenius problem consists in finding formulas for the Frobenius number and the genus of a numerical semigroup in terms of its minimal system of generators (see [5]). This problem is solved in [12] for numerical semigroups of embedding dimension equal to two, remaining open, in general, for numerical semigroups of embedding dimension greater than or equal to three.

Let T be a numerical semigroup. Given $d \in \mathbb{N} \setminus \{0\}$, we write

$$\frac{T}{d} = \{x \in \mathbb{N} \mid d\,x \in T\}.$$

In [7] it is proved that $\frac{T}{d}$ is a numerical semigroup. This semigroup is called the *quotient of T by d*. In this case, we also say that T is a *$d-$multiple of $\frac{T}{d}$*.

Correspondingly, given two numerical semigroups S and T, we say that T is a multiple of S if there exists $d \in \mathbb{N} \setminus \{0\}$ such that $\frac{T}{d} = S$; in particular, S is an arithmetic extension of T (see [4]).

This talk is devoted to the study of multiples of a numerical semigroup. The interest of this study is partially motivated by the problem that we present below. In [8] it is shown that a numerical semigroup is proportionally modular if and only if it has a multiple of embedding dimension two; this result together with those in [9] allow us to give an algorithmic procedure for deciding whether a numerical semigroup has a multiple of embedding dimension two. In [2] the problem of finding a numerical semigroup that has no multiples of embedding dimension three arises. In [3] the existence of such semigroups is proved although no concrete example is given. Recently, in [1, Theorem 3.1], examples of numerical semigroups that have no multiples of embedding dimension k are given for (fixed) $k \geq 3$.

Today it is still an open problem to give an algorithmic procedure that determines whether a given numerical semigroup has a multiple with embedding dimension greater than or equal to three. This problem is part of the more ambitious goal of determining

$$\min\{\mathrm{e}(T) \mid T \text{ is a multiple of } S\}$$

for a given numerical semigroup S. This number is called *quotient rank of* S (see [1]). It is clearly bounded above by $\mathrm{e}(S)$, since $S/1 = S$. Consequently, S is said to have *full quotient rank* when the quotient rank of S is equal to $\mathrm{e}(S)$.

Let S be a numerical semigroup and $d \in \mathbb{N} \setminus \{0\}$. We define

$$\mathrm{M}_d(S) := \left\{ T \in \mathscr{S} \mid \frac{T}{d} = S \right\}$$

and write $\max \mathrm{M}_d(S)$ for the set of maximal elements of $\mathrm{M}_d(S)$ with respect to the inclusion.

We will show that $\max \mathrm{M}_d(S)$ is finite and that the necessary and sufficient condition for S to be irreducible is that all elements of $\max \mathrm{M}_d(S)$ are irreducible (independently of d, in fact).

Moreover, we prove that there is a surjective map $\Theta_S^d : \mathrm{M}_d(S) \to \max \mathrm{M}_d(S)$. Thus, to compute $\mathrm{M}_d(S)$ it is enough to determine the fibers of Θ_S^d as long as $\max \mathrm{M}_d(S)$ can be computed following [6]. Although the number of fibers is finite, their cardinality is not necessarily finite (see [11]). So, we arrange the elements in each fiber of Θ_S^d is form of a rooted tree with root the corresponding element in $\max \mathrm{M}_d(S)$. This tree is locally finite, that is, the number of children of a given node is finite. Then, after characterizing the children of a given node, we can formulate a recursive process that will allow us to theoretically compute the elements of a fiber of Θ_S^d from the corresponding element in $\max \mathrm{M}_d(S)$. We also show that leaves (childless nodes) and fibers of cardinality one can appear. We emphasize that our recursive process can be easily truncated so as not to exceed some given Frobenius number or genus, thus producing a true algorithm.

Continuing with the study of the structure of $\mathrm{M}_d(S)$, we wil show that intersection of finitely many elements in $\mathrm{M}_d(S)$ is an element of $\mathrm{M}_d(S)$; this is not true for infinitely many elements, giving rise the notion of $\mathrm{M}_d(S)-$monoid. We prove that this particular family of submonoids of $(\mathbb{N}, +)$ are precisely those generated by $\{\langle X \rangle + d\,S \mid X \subset S, X \text{ is finite and } \langle X \rangle \cap (d(\mathbb{N} \setminus S)) = \varnothing\}$; moreover, if $\gcd(X \cup \{d\}) = 1$, then a numerical semigroup is obtained, and vice versa. Then, given a $\mathrm{M}_d(S)-$monoid, M, we prove there exists a unique $X \subset S$ such that $M = \langle X \rangle + d\,S$ and $M \neq \langle Y \rangle + dS$ for every $Y \subsetneq X$, and we introduce the notion of the $\mathrm{M}_d(S)-$embedding dimension of M as the cardinality of X.

Now, we will focus our attention on the elements of $\mathrm{M}_d(S)$ of $\mathrm{M}_d(S)-$embedding dimension one, we prove that these are a kind of generalization of a gluing of S and

$\mathbb{N}$. (see [10, Chapter 8] for more details on gluings). We solve the Frobenius problem for this family of numerical semigroups, as well as providing formulas for their pseudo-Frobenius numbers and type.

Finally, using some results in [1], we will derive a sufficient condition for a numerical semigroup to have full quotient rank. We will also give new examples of families with this property.

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