GLUINGS AND FIBERED SUMS

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I will talk about the existence of gluings and its relation with fibered sums. This is a joint work with C-Y. Jean Chan and Jung-Chen Liu. Our interests on the subject began by extending the relative viewpoint from numerical semigroups to affine semigroups.

An affine semigroup ring is usually regarded as a finitely generated algebra over a field κ. Besides such an absolute viewpoint, an affine semigroup ring can also be regarded as an algebra over another ring obtained from a subsemigroup. The latter perspective is what we refer to as the relative viewpoint. Previous works along this line have been conducted by Kim, Jafari and myself [8, 9, 10] in one-dimensional cases, that is, for numerical semigroup rings.

The current work is motivated by a question posed by Gimenez and Srinivasan [3] regarding when and how affine semigroup rings can be glued. For numerical semigroup rings, gluing has been extensively studied. It appeared in Watanabe [16, Lemma 1] in a special case following the study of complete intersection numerical semigroup rings by Herzog [7]. Delorme [1] treated gluing systematically for numerical semigroups and obtained a thorough characterization for complete intersections. Gluing was generalized to affine semigroup rings by Rosales [13]. It was further studied in Gimenez and Srinivasan [3, 4].

Recall that an *affine semigroup* S is a finitely generated monoid that is cancellative and torsion free. It can be embedded in $\mathbb{Z}^d$ for some positive integer d. Fixed an embedding, we may identify each element $s \in S$ with a d-tuple $(s_1, \ldots, s_d) \in \mathbb{Z}^d$. Let κ be a field. With respect to the embedding, the affine semigroup ring $\kappa[\mathbf{u}^S]$ is the κ-subalgebra of the Laurent polynomial ring $\kappa[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ generated by monomials of the form $u_1^{s_1} \cdots u_d^{s_d}$, where $s = (s_1, \ldots, s_d) \in S$. With the variables $u_1, \ldots, u_d$, our notation keeps track of the embedding, which is crucial to understand the nature of gluings. For example, an embedding of S may give a ring $\kappa[\mathbf{u}^S]$ that can not be glued to itself. On the other hand, two other embeddings of S may give two rings $\kappa[\mathbf{v}^S]$ and $\kappa[\mathbf{w}^S]$ that can be glued. Furthermore, changing of variables from $\mathbf{u}$ to $\mathbf{v}$ or from $\mathbf{u}$ to $\mathbf{w}$ may involve monomials with rational numbers as exponents.

The goal of gluing is to construct a new affine semigroup ring from two given ones by identifying selected subrings. There are two approaches: from set-theoretic viewpoint or from categorical viewpoint. The set-theoretic approach is the classical method initiated in Rosales [13], in which the part to be identified is a one-dimensional polynomial ring. To assure that no extra identification occurs outside the polynomial ring, this classical approach takes into account the defining relations of the given affine semigroup rings and those of the new one. Precisely, the new affine semigroup ring, if constructible, is restricted to acquire only one extra relation among the generators in addition to the existing relations in the given affine semigroup rings.

From another standpoint, our approach to obtain a new affine semigroup ring is categorical, where an affine semigroup ring is regarded as an object equipped with morphisms in the category of affine semigroup rings. An affine semigroup ring containing the two given ones with identified parts fits in a commutative diagram. I will talk about a particular diagram, smallest in the sense that all such diagrams emerge from it. So what we construct is a universal object among these diagrams, or in other words, a fibered sum in the category of affine semigroup rings. Our construction will be compared to the tensor product and to gluings of affine semigroup rings. While gluings of affine semigroup rings do not always exist, fibered sum can always be achieved. We investigate when the fibered sum of affine semigroup algebras gives rise to a gluing. A criterion for the existence of gluing is recovered.

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