THE FAR-FLUNG GORENSTEIN PROPERTY FOR NUMERICAL SEMIGROUPS

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Abstract. Extended abstract for IMNS 2024.

The stratification of the space between Cohen-Macaulay rings and Gorenstein rings has been an active area of research. Various classes of rings have been introduced (almost Gorenstein, nearly Gorenstein, quasi-Gorenstein, generalized Gorenstein etc.), and some of these concepts were motivated by the study of numerical semigroups. The size of Hilbert coefficients, trace ideals or the entries in the Betti table distinguished these properties.

In [1] we introduced the far-flung Gorenstein property for a 1-dimensional Cohen-Macaulay ring R with a canonical module ω_R. Recall that the trace of the canonical module is the ideal $\operatorname{tr}(\omega_R) = \sum_\varphi \operatorname{Im}\varphi \subseteq R$, where the sum is taken over all R-linear maps $\varphi : \omega_R \to R$. Set $\overline{R}$ the integral closure of R and $Q(R)$ the total ring of fractions of R. The conductor of the extension $R \subseteq \overline{R}$ is $R :_{Q(R)} \overline{R} = \{x \in Q(R) : x\overline{R} \subseteq R\}$. It is known that $\operatorname{tr}(\omega_R) \supseteq R :_{Q(R)} \overline{R}$.

Definition 0.1. ([1]) With notation as above, the ring R is called *far-flung Gorenstein* if

$$\operatorname{tr}(\omega_R) = R :_{Q(R)} \overline{R}.$$

Several ring theoretical properties of far-flung Gorenstein rings are studied in [1]. In particular, one has the following.

Theorem 0.2. *([1]) If R is a far-flung Gorenstein ring with Cohen-Macaulay type $r(R)$ and multiplicity $e(R)$, then*

$$r(R) + 1 \leq e(R) \leq \binom{r(R)+1}{2}.$$

The lower bound characterizes in fact when the ring R has minimal multiplicity, but the upper bound may not always be sharp.

Our focus is to describe the far-flung Gorenstein property for the (completion of the) semigorup ring $R = K[|H|]$ when H is a numerical semigroup. In the following let $e(H)$ be the multiplicity of H and $F(H)$ its Frobenius number. It is known that $K[|H|]$ is a one-dimensional Cohen-Macaulay domain and that the pseudo-Frobenius numbers $PF(H) = \{x \in \mathbb{Z} \setminus H : x + h \in H \text{ for all } 0 \neq h \in H\}$ determine the canonical module $\omega_R = (t^{-\alpha} : \alpha \in PF(H))R$. Moreover, the type of H, defined

2020 *Mathematics Subject Classification.* 13H10.

as $r(H) = |PF(H)|$, equals the Cohen-Macaulay type of $K[|H|]$. It is sometimes convenient to work with an isomorphic copy of ω_R, namely with

$$C = (t^{F(H)-\alpha} : \alpha \in PF(H))R$$

This has the advantage that $R \subseteq C \subseteq \overline{R} = K[|t|]$.

We obtain a first characterization (valid not just for semigroup rings).

Theorem 0.3. *([1]) The ring $K[|H|]$ is far-flung Gorenstein if and only if $C^2 = K[|H|]$.*

Specific to semigroup rings we get

Theorem 0.4. *([1]) Let H be a numerical semigroup and K any field. The following statements are equivalent:*

(1) the ring $K[|H|]$ is far-flung Gorenstein;
(2) $\{0, \ldots, e(H) - 1\} \subseteq \{2F(H) - \alpha - \beta : \alpha, \beta \in \mathrm{PF}(H)\}$;
(3) $\{2F(H) - e(H) + 1, \ldots, 2F(H)\} \subseteq \{\alpha + \beta : \alpha, \beta \in \mathrm{PF}(H)\}$.

Consequently, the far-flung property of $K[|H|]$ depends on H and not on the field K.

Corollary 0.5. *([1]) Let H be a numerical semigroup minimally generated by $a_1 < \cdots < a_v$ which is of minimal multipilcity, i.e. $v = a_1$. Then $K[|H|]$ is a far-flung Gorenstein ring if and only if*

$$\{2a_v - a_1 + 1, \ldots, 2a_v\} \subseteq \{a_i + a_j : 2 \leq i, j \leq v\}.$$

We thus obtain an infinite family of far-flung numerical semigroups.

Corollary 0.6. *Let $a \geq 3$ and d be coprime nonnegative integers and $H = \langle a, a + d, \ldots, a + (a-1)d\rangle$. Then $R = K[|H|]$ is a far-flung Gorenstein ring if andy only if $d = 1$.*

Problem 0.7. (The Rohrbach problem, [3, 4]) Let A be a set of non-negative integers with r elements. Let $n(A)$ denote the integer such that the sum-set

$$A + A = \{a + b : a, b \in A\}$$

contains the integers $0, 1, \ldots, n(A) - 1$ but not $n(A)$. If $0 \notin A$ then let $n(A) = -1$. For $r > 0$ the Rohrbach problem asks to find the integer

$$\overline{n}(r) = \max\{n(A) : |A| = r\}.$$

With this notation we have the following result, which improves Theorem 0.2.

Corollary 0.8. *Suppose that R is a numerical semigroup ring. If R is a far-flung Gorenstein ring, then the inequality*

$$e(R) \leq \overline{n}(r)$$

holds, where $r = r(R)$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13
$\overline{n}(r)$	1	3	5	9	13	17	21	27	33	41	47	55	65
r	14	15	16	17	18	19	20	21	22	23	24	25	$\cdots$
$\overline{n}(r)$	73	81	93	105	117	129	141	153	165	181	197	213	$\cdots$

The solution $\overline{n}(r)$ of the Rohrbach problem is known for $r \leq 25$ (see [2, 4]):

The number $\overline{n}(r)$ provides the sharp upper bounds of the multiplicity of far-flung Gorenstein numerical semigroup rings, if $r = \mathrm{r}(R) \leq 5$. Note that already when $r = 3$ one has $5 = \overline{n}(3) < \binom{3+1}{2} = 6$, hence the solution to the Rohrbach problem gives a better bound than the one in Theorem 0.2. The question remains whether there exist far-flung Gorenstein numerical semigroups H of type r where $e(H) = \overline{n}(r)$, for any r.

When the type of R is small, one gets a full classification.

Proposition 0.9. *Let $R = K[|H|]$ be a numerical semigroup ring with $r(R) = 2$. Then R is far flung Gorenstein if and only if $H = \langle 3, 3n+1, 3n+2\rangle$ for some integer $n > 0$.*

Theorem 0.10. *Let $R = K[|H|]$ be a numerical semigroup ring. Suppose that R is of type 3 and not of minimal multiplicity. Then R is a far-flung Gorenstein ring if and only if H is in one of the following families of semigroups*

(1) $H = \langle 5, 5m + 4, 10m + 6, 10m + 7\rangle$, where $m \geq 1$;
(2) $H = \langle 5, 5m + 1, 10m + 3, 10m + 4\rangle$, where $m \geq 1$;
(3) $H = \langle 5, 5m + 2, 10m + 1, 10m + 3\rangle$, where $m \geq 1$;
(4) $H = \langle 5, 5m + 3, 10m + 4, 10m + 7\rangle$, where $m \geq 1$.

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