GENERALIZED NUMERICAL SEMIGROUPS UP TO PERMUTATIONS OF COORDINATES AND SOME RELATED PROCEDURES

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Let $\mathbb{N}$ be the set of non negative integers and d be a positive integer. A *generalized numerical semigroup* (GNS) is a submonoid S of $\mathbb{N}^d$ such that the set $\mathrm{H}(S) = \mathbb{N}^d \setminus S$ is finite. The set $\mathrm{H}(S)$ is called the set of *gaps* (or *holes*) of S and its cardinality $\mathrm{g}(S) = |\,\mathrm{H}(S)|$ is called the *genus* of S. Generalized numerical semigroups have been introduced in [2] as a straightforward generalization of the well known concept of numerical semigroup, that is a submonoid of $\mathbb{N}$ having finite complement in it. Some notions and results concerning numerical semigroups have been generalized to the context of GNSs in some recent papers. We recall the definition of *relaxed monomial order* (introduced in [2]): it is a total order $\preceq$ in $\mathbb{N}^d$ satisfying the two properties: *a*) $\mathbf{0} \prec \mathbf{v}$ for all $\mathbf{v} \in \mathbb{N}^d \setminus \{\mathbf{0}\}$; *b*) for all $\mathbf{v}, \mathbf{w} \in \mathbb{N}^d$ such that $\mathbf{v} \preceq \mathbf{w}$ then $\mathbf{v} \preceq \mathbf{w} + \mathbf{u}$ for all $\mathbf{u} \in \mathbb{N}^d$. Moreover, for a GNS $S \neq \mathbb{N}^d$ and a fixed relaxed monomial order $\preceq$, we define $\mathbf{F}_{\preceq}(S) = \max_{\preceq}(\mathrm{H}(S))$ and $\mathbf{m}_{\preceq}(S) = \min_{\preceq}(S \setminus \{0\})$.

In our work, we consider an equivalence relation $\simeq$ in the set $\mathcal{S}_d$ of all generalized numerical semigroups in $\mathbb{N}^d$, where $S_1 \simeq S_2$ if S_2 can be obtained by a permutation of the coordinates of S_1. It easy to see that if $S_1 \simeq S_2$, then S_1 and S_2 are isomorphic as monoids. We show that the converse holds, that is, if S_1 and S_2 are isomorphic GNSs in $\mathbb{N}^d$, then $S_1 \simeq S_2$. Furthermore, let $\mathcal{S}_{g,d}$ be the set of all generalized numerical semigroups in $\mathbb{N}^d$ having fixed genus g. Two different procedures to compute this set are provided in [1, 2]. In particular, they are associated to the constructions of some rooted tree graphs, depending on a fixed relaxed monomial order $\preceq$, related to the following transformations defined in the set $\mathcal{S}_d \setminus \{\mathbb{N}^d\}$: $\mathcal{J}_{\preceq} : S \mapsto S \cup \{\mathbf{F}_{\preceq}(S)\}$ and $\mathcal{O}_{\preceq} : S \mapsto (S \cup \{\mathbf{F}_{\preceq}(S)\}) \setminus \{\mathbf{m}_{\preceq}(S)\}$. A natural question is how it is possible to "trim" these rooted trees in order to only one semigroup in $[S]_{\simeq}$, for every $S \in \mathcal{S}_{g,d}$, appears in the "trimmed trees". To this aim, we show a possible criterion to choice a representative in the equivalence class $[S]_{\simeq}$, for every $S \in \mathcal{S}_{g,d}$. Using this criterion, we define a set $\mathrm{R}_{\preceq}(\mathcal{S}_d)$, containing only these representatives, having the property that the restrictions of the transforms $\mathcal{J}_{\preceq}$ and $\mathcal{O}_{\preceq}$ in this set are well defined, that is, if $S \in \mathrm{R}_{\preceq}(\mathcal{S}_d) \setminus \{\mathbb{N}\}^d$, then $\mathcal{J}_{\preceq}(S), \mathcal{O}_{\preceq}(S) \in \mathrm{R}_{\preceq}(\mathcal{S}_d)$. Some properties related to our choice of the representatives are investigated. Moreover, these constructions allows to define possible procedures to generate the set of all GNSs of given genus up to permutations of coordinates, that is, producing only one semigroup for every equivalence class of $\simeq$.

This is joint work with Gioia Failla and Francesco Navarra.

References

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